

Group Theory of Non-Abelian Vortices

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Abstract

We investigate the structure of the moduli space of multiple BPS non-Abelian vortices in $U(N)$ gauge theory with N fundamental Higgs fields, focusing our attention on the action of the exact global (color-flavor diagonal) $SU(N)$ symmetry on it. The moduli space of a single non-Abelian vortex, $\mathbb{C}P^{N-1}$, is spanned by a vector in the fundamental representation of the global $SU(N)$ symmetry. The moduli space of winding-number k vortices is instead spanned by vectors in the direct-product representation: they decompose into the sum of irreducible representations each of which is associated with a Young tableau made of k boxes, in a way somewhat similar to the standard group composition rule of $SU(N)$ multiplets. The Kähler potential is exactly determined in each moduli subspace, corresponding to an irreducible $SU(N)$ orbit of the highest-weight configuration.

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1 Introduction and discussion

Non-Abelian vortices have been discovered several years ago in the context of $U(N)$ supersymmetric gauge theories and in string theory [1, 2]. BPS non-Abelian vortices exist in the $U(N)$ Yang-Mills theory coupled to $N_F = N$ Higgs fields in the fundamental representation. The BPS equations are of the form

$$(\mathcal{D}_1 \pm i\mathcal{D}_2)H = 0, \quad F_{12} = \pm \frac{g^2}{2} (HH^\dagger - v^2 \mathbf{1}_N), \quad (1.1)$$

where the upper (lower) sign describes the vortices (anti-vortices). The Higgs fields H are combined in a color-flavor mixed $N \times N$ matrix on which the $U(N)$ gauge (color) symmetry acts on the left while the $SU(N)$ flavor symmetry acts on the right. The constant g is the $U(N)$ gauge coupling¹ and v^2 is the Fayet-Iliopoulos parameter. The $U(N)_C$ gauge (color) symmetry is spontaneously broken completely in the so-called color-flavor-locked vacuum ($\langle H \rangle = v \mathbf{1}_N$), whereas the global diagonal symmetry $SU(N)_{C+F}$ remains unbroken. Since winding-number k vortices (“ k vortices” from now on, for simplicity) saturate the BPS energy (tension) bound

$$T \geq 2\pi v^2 k, \quad (1.2)$$

no net forces are exerted among the static vortices. This implies that a set of solutions to Eq. (1.1) contains integration constants, i.e. moduli parameters parametrizing the set of configurations with degenerate energy, viz. the moduli space of BPS vortices: \mathcal{M}_k .

In addition to the position moduli, each non-Abelian vortex has internal orientational moduli which are associated with the $SU(N)_{C+F}$ color-flavor symmetry, broken by the individual vortex configurations. Consider for instance a particular BPS solution

$$H = \text{diag} (H^{\text{ANO}}, v, \dots, v), \quad A_\mu = \text{diag} (A_\mu^{\text{ANO}}, 0, \dots, 0), \quad (1.3)$$

where H^{ANO} and A_μ^{ANO} are the fields describing the well-known Abrikosov-Nielsen-Olesen (ANO) vortex solution. Clearly, the solution breaks $SU(N)_{C+F}$ down to $SU(N-1) \times U(1)$ and therefore the corresponding Nambu-Goldstone zero-modes, which we call internal orientational modes, appear on the vortex and parametrize the coset

$$\frac{SU(N)}{SU(N-1) \times U(1)} \cong \mathbb{C}P^{N-1}, \quad (1.4)$$

¹ Here we take common gauge couplings for $SU(N)$ and $U(1)$ of $U(N) = [U(1) \times SU(N)]/\mathbb{Z}_N$ for simplicity.

whose size (Kähler class) is given by $4\pi/g^2$ [3–6]. The generic vortex solutions can be obtained by acting on the above solution with $U \in SU(N)_{\text{C+F}}$, i.e., $H \rightarrow U^\dagger H U$, $A_\mu \rightarrow U^\dagger A_\mu U$. We parametrize them by using a normalized complex N -vector $\vec{\phi}$ ($\vec{\phi}^\dagger \cdot \vec{\phi} = 1$)

$$H = v\mathbf{1}_N + (H^{\text{ANO}} - v)\vec{\phi}\vec{\phi}^\dagger, \quad A_\mu = A_\mu^{\text{ANO}}\vec{\phi}\vec{\phi}^\dagger. \quad (1.5)$$

Since the overall $U(1)$ phase of $\vec{\phi}$ is unphysical, the vector $\vec{\phi}$ can indeed be interpreted as the homogeneous coordinates of \mathbb{CP}^{N-1} . The moduli space of multiple vortices was found in Ref. [7] in terms of the moduli matrix formalism [8,9]. The moduli space metric has been found recently for well-separated vortices [10,11] using a generic formula for the Kähler potential on the moduli space [12].

The starting point of our analysis is the observation that the vector $\vec{\phi}$ for a single vortex transforms according to the fundamental representation of $SU(N)_{\text{C+F}}$. In this precise sense *the non-Abelian vortex belongs to the fundamental representation of $SU(N)_{\text{C+F}}$* . Now the following question naturally arises: *How does the color-flavor symmetry act in the moduli space \mathcal{M}_k , and to which representations do k vortices belong?* Since each vortex has an orientational vector $\vec{\phi}_I$ ($I = 1, \dots, k$) in the fundamental representation of $SU(N)_{\text{C+F}}$, one expects that it is simply described by the tensor product of fundamental representations, e.g.,

$$\square \otimes \square = \square\square \oplus \begin{smallmatrix} \square \\ \square \end{smallmatrix}. \quad (1.6)$$

However, the situation is not so simple since the orientational vectors $\vec{\phi}_I$ are well-defined only when all vortices are separated. What happens when two or more vortices sit on top of each other? To answer these questions we must study the moduli space in such a way that allows a smooth limiting case where the vortex centers are taken to be coincident.

The problem was already studied in the literature for $k = 2$ coincident vortices in $U(2)$ gauge theory [13–18], and partial answers were obtained. While each vortex carries an orientation in \mathbb{CP}^1 , the moduli space of two coincident vortices was found to be $W\mathbb{CP}_{(2,1,1)}^2 \simeq \mathbb{CP}^2/\mathbb{Z}_2$ [13–15]. In this case, each vortex belongs to **2** so that the composition rule (1.6) yields $\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$. We have indeed found the moduli parameters transforming as **3** and a singlet configuration corresponding to a \mathbb{Z}_2 singularity [15]. However, the precise knowledge about the correspondence between the representations and points in the moduli space was lacking. In other words, we did not know the true meaning of the composition-decomposition Eq. (1.6) at that time.

These questions are clarified in the present paper.

These issues are actually intimately related to the question of the non-Abelian *monopoles*. Indeed, a $U(N)$ vortex system such as ours can always be regarded as a low-energy approximation of an underlying larger, e.g. $SU(N+1)$, gauge theory, spontaneously broken to $SU(N) \times U(1)$ gauge group, by the vacuum expectation value (VEV) of some other scalar field at a mass scale much higher than the typical vortex mass scale. In such a hierarchical symmetry-breaking setting, whatever properties we find out about the vortices can be translated into those of the massive monopoles sitting at the extremes, as a homotopy-sequence consideration relates the two, at least semi-classically [17]². We shall, however, not dwell much on these points in the present work: we shall come back to them elsewhere.

In this paper the moduli space of k vortices are studied by using the $U(k)$ Kähler quotient construction due to Hanany-Tong [1]. We analyze the moduli space in algebraic geometry by using certain $SL(k, \mathbb{C})$ invariants: symmetric polynomials of the vortex centers and “baryonic invariants”³. We find algebraic constraints for these invariants which specify the embedding of the internal moduli space in a complex projective space. The moduli space of vortices contains various $SU(N)$ orbits, each of which belongs to a certain representation of $SU(N)$. We analyze the structures of those $SU(N)$ orbits by using “vortex state vectors” constructed from the $SL(k, \mathbb{C})$ invariants, by the help of some auxiliary harmonic-oscillator states.

When k vortices are all separated, vortex states can be written as coherent states in such a description. Accordingly, the vortex states can be shown to correspond to factorized (non-entangled) products of k single vortex states in the fundamental representation.

The situation of the k -winding vortices with coincident centers turns out to be considerably subtler. It will be shown that each $SU(N)$ orbit of k rotationally invariant (axially symmetric) vortices corresponding to some irreducible representation, which we call the “irreducible $SU(N)$ orbit,” can be classified by a Young tableau with k boxes. Generic orbits belong to reducible

²The monopole-vortex correspondence becomes far subtler when one is interested in the properties of light monopoles. The low-energy dynamics and renormalization-group effects both for the vortex [3,4] and monopole [19] must be properly taken into account. This requires also a careful identification of the quantum vacua [20], as many of the systems involved possess large *vacuum moduli*.

³Although they have nothing to do with real physical baryons, for formal similarity and for convenience these invariants will be referred to “baryonic invariants” or simply as “baryons”: see Section 2.1 below.

representations and the associated vortex states can be written as a superposition of irreducible states.

One of the deepest aspects of our results is the fact that the vortex moduli, which describe a degenerate set of classical extended field configurations, behave under the exact $SU(N)$ global symmetry as a moduli space of quantum oscillator states, characterized by irreducible multiplets and having the possibility of superposition of “states”. Even if this should be regarded just as a formal aspect of mathematical interest here, it could provide a physical key to quantum-mechanical understanding of non-Abelian monopoles through the vortex-monopole connection, briefly mentioned above.

Also, albeit our results here – understandably – basically obey the standard composition rule for $SU(N)$ multiplets, the composition rule of the non-Abelian vortices is found to possess various special features (see below); for instance, the vortex moduli space involves in general much fewer dimensions than naïvely expected.

All irreducible $SU(N)$ orbits are Kähler submanifolds inside the full moduli space. We shall construct the Kähler potential on each of the irreducible $SU(N)$ orbits and find that the coefficient of the Kähler potential is quantized as an integer: the latter is uniquely specified by the associated Young tableau. We point out the existence of a duality between pairs of irreducible orbits corresponding to the conjugate representations of $SU(N)$, which are found to describe, as expected, the same low-energy effective action.

The rest of the paper is organized as follows. In Section 2, the basic features of the moduli space of k non-Abelian vortices are reviewed. We then proceed to construct the “baryonic invariants” which form good coordinates on our moduli space. By making use of these we find the representations of k separated vortices in Section 2.2; we construct an irreducible representation for a specific (highest-weight) configuration of coincident vortices in Section 2.3. In Section 3 the solution to the constraints on the “baryonic invariants” is worked out and the result is used to show the $SU(N)$ decomposition rule for generic vortex solutions for given k . A particular attention is paid to the consideration of the limit of co-axial vortices. The cases of $k = 1, 2, 3$ are explicitly solved, while a general recipe is given, valid for any N, k . The Kähler potentials for the irreducible $SU(N)$ orbits are obtained in Section 4. A brief summary and outlook is given in Section 5. A few details of our analysis are postponed to the Appendices.

2 Moduli space of non-Abelian vortices

2.1 The moduli space and $GL(k, \mathbb{C})$ invariants

The moduli space of the non-Abelian $U(N)$ vortices governed by the BPS Eq. (1.1) was first studied by Hanany-Tong [1]. There the dimension of the moduli space \mathcal{M}_k of k vortices has been shown by using an index theorem calculation to be⁴

$$\dim_{\mathbb{C}} \mathcal{M}_k = kN, \quad (2.1)$$

with k being the topological winding number. Moreover, they found a D-brane configuration and derived a Kähler quotient construction for \mathcal{M}_k . It is sometimes called a *half-ADHM construction* by analogy with the moduli space of instantons. In the D-brane configuration, the k vortices are k D2-branes suspended between N D4-branes and an NS5-brane. The low-energy effective field theory on the k D2-branes is described by a $U(k)$ gauge theory coupled with a k -by- k matrix Z in adjoint representation and a k -by- N matrix ψ in the fundamental representation \mathbf{k} of the $U(k)$ gauge symmetry, given by D2–D2 strings and D2–D4 strings, respectively. The $U(k)$ gauge symmetry on the D2-branes acts on Z and ψ as

$$(Z, \psi) \rightarrow (gZg^{-1}, g\psi), \quad g \in U(k). \quad (2.2)$$

The moduli space \mathcal{M}_k can be read off as the Higgs branch of vacua in the $U(k)$ gauge theory on the k D2-branes, which is the Kähler quotient of the $U(k)$ action (2.2)⁵

$$\mathcal{M}_k \cong \mathcal{M}_k^{\text{HT}} \equiv \{ (Z, \psi) \mid \mu_D = r\mathbf{1}_k \} / U(k), \quad (2.3)$$

$$\mu_D \equiv [Z, Z^\dagger] + \psi\psi^\dagger. \quad (2.4)$$

This Kähler quotient gives a natural metric on \mathcal{M}_k provided that (Z, ψ) has a flat metric on $\mathbb{C}^{k(k+N)}$. Unfortunately, the geodesics of such a metric do not describe the correct dynamics of vortices [1]. The 2d FI parameter r is related to the 4d gauge coupling constant by

$$r = \frac{4\pi}{g^2}, \quad (2.5)$$

⁴The general result of Ref. [1] in $U(N)$ theory for $N_F \geq N$ flavors is $\dim_{\mathbb{C}} \mathcal{M}_k = kN_F$. However, we restrict our attention to the case $N_F = N$ and hence local vortices in this paper.

⁵ Here the normalization of the scalar fields Z, ψ is chosen so that they have canonical kinetic terms in the 2-dimensional effective gauge theory on the D2 branes. In this convention the eigenvalues of Z (i.e. vortex positions) are dimensionless parameters.

which holds under the RG flow if the 4d theory has $\mathcal{N} = 2$ supersymmetry and the 2d theory has $\mathcal{N} = (2, 2)$ supersymmetry [3, 4].

According to Ref. [21] the Kähler quotient (2.3) can be rewritten as a complex symplectic quotient as

$$\mathcal{M}_k \cong \{ (Z, \psi) \} // GL(k, \mathbb{C}), \quad (2.6)$$

where instead of having the D -term condition $\mu_D = r\mathbf{1}_k$, the pair of matrices (Z, ψ) is divided by the complexified non-compact group $U(k)^\mathbb{C} = GL(k, \mathbb{C})$ which acts in the same way as Eq. (2.2). Here the quotient denoted by the double slash “//” means that points at which the $GL(k, \mathbb{C})$ action is not free should be removed so that the group action is free at any point. This quotient is also understood as the algebro-geometric quotient, so that the quotient space is parametrized by a set of $GL(k, \mathbb{C})$ holomorphic invariants with suitable constraints, see e.g. Ref. [22].

The starting point of our analysis, Eq. (2.6), can also be obtained directly from a purely field-theoretic point of view, based on the BPS equation (1.1). It has been shown by using the moduli-matrix approach [7–9] that all the moduli parameters of the k -vortex solutions are summarized exactly as in Eq. (2.6). The 4d field theory also provides the correct metric on \mathcal{M}_k describing the dynamics of vortices as a geodesic motion on the moduli space. Although a general formula for the metric and its Kähler potential has been derived [12], the explicit form of the metric is however difficult to obtain since no analytic solutions to the BPS equation are known. Nevertheless, the asymptotic metric for well-separated vortices has recently been found in Ref. [10].⁶

In what follows, we analyze the moduli space Eq. (2.6) without assuming any metric *a priori*. Our prime concern is how the exact global $SU(N)$ symmetry acts on the vortex moduli space \mathcal{M}_k . The matrix Z is a singlet while ψ belongs to the fundamental representation \mathbf{N} . Namely, the $SU(N)$ acts on Z and ψ as

$$Z \rightarrow Z, \quad \psi \rightarrow \psi \mathcal{U}, \quad \mathcal{U} \in SU(N). \quad (2.7)$$

As will be seen this action induces a natural $SU(N)$ action on the moduli space of vortices. We will also discuss the metrics on the symmetry orbits on which the $SU(N)$ acts isometrically. To this end, we use the algebro-geometric construction [22] of the moduli space by using the $GL(k, \mathbb{C})$ invariants which provide a set of coordinates of the moduli space.

⁶ See Ref. [11] for an alternative formula for vortices on Riemann surfaces.

Clearly, the coefficients σ_i ($i = 1, \dots, k$) of the characteristic polynomial of Z are invariants of the $GL(k, \mathbb{C})$ action

$$\det(\lambda \mathbf{1}_k - Z) = \lambda^k + \sum_{i=1}^k (-1)^i \sigma_i \lambda^{k-i}. \quad (2.8)$$

Since the vortex positions z_I ($I = 1, \dots, k$) are defined as the eigenvalues of Z (roots of the characteristic polynomial)

$$\det(\lambda \mathbf{1}_k - Z) = \prod_{I=1}^k (\lambda - z_I), \quad (2.9)$$

the parameters σ_i and z_I are related by

$$\sigma_i = P_i(z_1, \dots, z_k), \quad (2.10)$$

where P_i ($i = 1, \dots, k$) are the elementary symmetric polynomials defined by

$$P_i(z_1, \dots, z_k) \equiv \sum_{1 \leq I_1 < \dots < I_i \leq k} z_{I_1} z_{I_2} \dots z_{I_i}. \quad (2.11)$$

Note that vortex positions z_I are not fully invariant under $GL(k, \mathbb{C})$ transformations since they can be exchanged by the Weyl group \mathfrak{S}_k .

Other invariants can be constructed as follows. Let $Q^{(n)}$ ($n = 0, 1, \dots$) be the following (\mathbf{k}, \mathbf{N}) matrices of $SL(k, \mathbb{C}) \times SU(N)$ (Eqs. (2.2) and (2.7)):

$$Q^{(0)} \equiv \psi, \quad Q^{(1)} \equiv Z\psi, \quad \dots, \quad Q^{(n)} \equiv Z^n \psi, \quad \dots. \quad (2.12)$$

One can construct $SL(k, \mathbb{C}) \subset GL(k, \mathbb{C})$ invariants from $Q^{(n)}$ by using the totally anti-symmetric tensor $\epsilon^{i_1 \dots i_k}$ as

$$B_{r_1 r_2 \dots r_k}^{n_1 n_2 \dots n_k} \equiv \epsilon^{i_1 i_2 \dots i_k} Q_{i_1 r_1}^{(n_1)} Q_{i_2 r_2}^{(n_2)} \dots Q_{i_k r_k}^{(n_k)}. \quad (2.13)$$

We call these the “baryonic invariants” or sometimes simply “the baryons” below, relying on a certain analogy to the baryon states in the quark model (or in quantum chromodynamics).

Remark: although obviously they have no physical relation to the real-world baryons (the proton, neutron, etc.), no attentive reader should be led astray by such a short-hand notation.

Note that the baryons (2.13) are invariant under $SL(k, \mathbb{C})$ and transform under the remaining $U(1)^\mathbb{C} \cong \mathbb{C}^*$ as

$$B_{r_1 r_2 \dots r_k}^{n_1 n_2 \dots n_k} \rightarrow e^\lambda B_{r_1 r_2 \dots r_k}^{n_1 n_2 \dots n_k}, \quad (2.14)$$

with a suitable weight λ .

The vortex positions $\{z_I\} \cong \mathbb{C}^k / \mathfrak{S}_k \cong \mathbb{C}^k$ are parametrized by the moduli parameters $\{\sigma_i\} \cong \mathbb{C}^k$. In addition to these parameters, there are baryons

$$\{B_{r_1 r_2 \dots r_k}^{n_1 n_2 \dots n_k}\} \cong V,$$

as moduli parameters, where V denotes an infinite-dimensional complex linear space spanned by the baryons. The problem is that not all of these invariants are independent of each other; the baryons $B_{r_1 r_2 \dots r_k}^{n_1 n_2 \dots n_k}$ and σ_i satisfy certain constraints by construction. Therefore, the vortex moduli space Eq. (2.6) can be rewritten as

$$\mathcal{M}_k \cong \{ \mathbb{C}^k \times V \mid \text{constraints} \} // \mathbb{C}^*. \quad (2.15)$$

Since the baryonic invariants transform under $SU(N)$, there exists a linear action of $SU(N)$ on V : this induces an $SU(N)$ action on the moduli space.

Consider now the constraints on the parameters σ_i and the baryons $B_{r_1 r_2 \dots r_k}^{n_1 n_2 \dots n_k}$ in more detail. For this purpose it turns out to be convenient to introduce an auxiliary set of k linear harmonic oscillator states, each of which carrying an $SU(N)$ label, and make a map from the vector space V to the Fock space of such oscillators. Let us introduce a “vortex state vector”⁷ $|B\rangle \in V$ by

$$|B\rangle \equiv \sum_{\substack{n_1, n_2, \dots, n_k \\ r_1, r_2, \dots, r_k}} \frac{1}{(n_1! n_2! \dots n_k!)^{\frac{1}{2}}} B_{r_1 r_2 \dots r_k}^{n_1 n_2 \dots n_k} |n_1, r_1\rangle \otimes |n_2, r_2\rangle \otimes \dots \otimes |n_k, r_k\rangle, \quad (2.16)$$

with $n_i \in \mathbb{Z}_{\geq 0}$, $1 \leq r_i \leq N$; the associated annihilation and creation operators $\hat{a}_i, \hat{a}_i^\dagger$ ($i = 1, \dots, k$)

$$\hat{a}_i \left(\dots \otimes |n_i, r_i\rangle \otimes \dots \right) = \sqrt{n_i} \left(\dots \otimes |n_i - 1, r_i\rangle \otimes \dots \right), \quad (2.17)$$

$$\hat{a}_i^\dagger \left(\dots \otimes |n_i, r_i\rangle \otimes \dots \right) = \sqrt{n_i + 1} \left(\dots \otimes |n_i + 1, r_i\rangle \otimes \dots \right) \quad (2.18)$$

satisfy the standard commutation relations

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \quad [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0. \quad (2.19)$$

Note that once $|B\rangle$ is given, the baryonic invariants can be read off from the following relation

$$B_{r_1 r_2 \dots r_k}^{n_1 n_2 \dots n_k} = \langle 0, r_1; \dots; 0, r_k | (\hat{a}_1)^{n_1} (\hat{a}_2)^{n_2} \dots (\hat{a}_k)^{n_k} |B\rangle, \quad (2.20)$$

where $|0, r_1; \dots; 0, r_k\rangle \equiv |0, r_1\rangle \otimes \dots \otimes |0, r_k\rangle$ are the ground states. Now there are three types of constraints to be taken into account (see Appendix A for more details):

⁷We hasten to add that no relation between the notion of vortex “state vectors” here and any quantum dynamics is implied by such a construction.

1. From definition (2.13) one can see that the baryons satisfy the anti-symmetry property

$$B^{A_1 \cdots A_i \cdots A_j \cdots A_k} = -B^{A_1 \cdots A_j \cdots A_i \cdots A_k}, \quad (2.21)$$

where A_i stands for the pair of indices (n_i, r_i) . This constraint can be rewritten as

$$\hat{\rho} |B\rangle = \text{sign}(\rho) |B\rangle, \quad (2.22)$$

where $\hat{\rho}$ denotes an element of the symmetric group \mathfrak{S}_k . For an element $\hat{\rho} \in \mathfrak{S}_k$

$$\rho = \begin{pmatrix} 1 & 2 & \cdots & k \\ I_1 & I_2 & \cdots & I_k \end{pmatrix}, \quad (2.23)$$

the action on the state is defined by

$$\hat{\rho} |n_1, r_1\rangle \otimes |n_2, r_2\rangle \otimes \cdots \otimes |n_k, r_k\rangle = |n_{I_1}, r_{I_1}\rangle \otimes |n_{I_2}, r_{I_2}\rangle \otimes \cdots \otimes |n_{I_k}, r_{I_k}\rangle. \quad (2.24)$$

2. The second condition is a consequence of the relation $Q^{(n+m)} = Z^m Q^{(n)}$. It follows that

$$P_i(\hat{a}_1, \cdots, \hat{a}_k) |B\rangle = \sigma_i |B\rangle, \quad (i = 1, \dots, k), \quad (2.25)$$

where $P_i(\hat{a}_1, \cdots, \hat{a}_k)$ are the elementary symmetric polynomials made of \hat{a}_i (cfr. Eq. (2.11)).

3. The last type of constraints are the quadratic equations for the baryons, which follow from Eq. (2.13):

$$B^{A_1 A_2 \cdots A_{k-1} [A_k} B^{B_1 B_2 \cdots B_k]} = 0, \quad (2.26)$$

where A_i stands for the pair of indices (n_i, r_i) . This constraint is a generalization of the Plücker relations for the Grassmannian.

Eqs. (2.22) and (2.25) can be viewed as linear constraints for baryons with σ_i -dependent coefficients. Therefore, for a given set of values $\{\sigma_i\}$, they define a linear subspace $W(\sigma_i) \subset V$ to which the vortex state vector $|B\rangle$ belongs. We will see that the representation of the $SU(N)$ action on $W(\sigma_i)$ is independent of σ_i and isomorphic to k copies of the fundamental representation \mathbf{N}

$$W(\sigma_i) \cong \mathbb{C}^{N^k} \cong \bigotimes_{i=1}^k \mathbf{N}. \quad (2.27)$$

Note that not all vectors in this “state space” $W(\sigma_i)$ represent vortex state vectors since they must still satisfy Eq. (2.26). Namely, the vortex moduli space is defined by the constraints (2.26), which are quadratic homogeneous polynomials of the coordinates of $W(\sigma_i)$ with σ_i -dependent coefficients.

2.2 The moduli space of k separated vortices

Let us first consider the case of winding-number k vortices with distinct centers, $z_I \neq z_J$ (for all $I \neq J$). It follows from Eq. (2.25) that for $i = 1, 2, \dots, k$

$$\prod_{I=1}^k (\hat{a}_i - z_I) |B\rangle = \left((\hat{a}_i)^k + \sum_{n=1}^k (-1)^n \sigma_n (\hat{a}_i)^{k-n} \right) |B\rangle = \prod_{j=1}^k (\hat{a}_i - \hat{a}_j) |B\rangle = 0. \quad (2.28)$$

Thus, in the case of $z_I \neq z_J$, there exists an I_i ($1 \leq I_i \leq k$) for each i such that⁸

$$\hat{a}_i |B\rangle = z_{I_i} |B\rangle. \quad (2.29)$$

Namely, the most generic form of the solution to the constraint (2.28) is

$$|B\rangle = \sum_{\substack{I_1, I_2, \dots, I_k \\ r_1, r_2, \dots, r_k}} \tilde{B}_{r_1 r_2 \dots r_k}^{I_1 I_2 \dots I_k} |z_{I_1}, r_1\rangle \otimes |z_{I_2}, r_2\rangle \otimes \dots \otimes |z_{I_k}, r_k\rangle, \quad (2.30)$$

where $|z_{I_i}, r_i\rangle$ are the coherent states defined by

$$|z_{I_i}, r_i\rangle \equiv \exp\left(z_{I_i} \hat{a}_i^\dagger\right) |0, r_i\rangle. \quad (2.31)$$

Recall that the coherent states are eigenstates of the annihilation operators

$$\hat{a}_i |z_{I_i}, r_i\rangle = z_{I_i} |z_{I_i}, r_i\rangle. \quad (2.32)$$

Then the constraint (2.25) reads

$$P_i(z_{I_1}, z_{I_2}, \dots, z_{I_k}) |B\rangle = \sigma_i |B\rangle \quad \left(= P_i(z_1, z_2, \dots, z_k) |B\rangle \right). \quad (2.33)$$

This means that $\{z_{I_1}, z_{I_2}, \dots, z_{I_k}\}$ is a permutation of $\{z_1, z_2, \dots, z_k\}$. Taking into account the anti-symmetry condition (2.22), the solution of the constraints (2.22) and (2.25) is given by

$$|B\rangle = \sum_{r_1, r_2, \dots, r_k} \tilde{B}_{r_1 r_2 \dots r_k} \hat{\mathcal{A}} \left(|z_1, r_1\rangle \otimes |z_2, r_2\rangle \otimes \dots \otimes |z_k, r_k\rangle \right), \quad (2.34)$$

where $\hat{\mathcal{A}}$ denotes the anti-symmetrization of the states

$$\hat{\mathcal{A}} \equiv \frac{1}{k!} \sum_{\rho \in \mathfrak{S}_k} \text{sign}(\rho) \hat{\rho}. \quad (2.35)$$

⁸Note that this relation does not necessarily hold for coincident vortices. For example, if $z_I = z_J = z_0$ ($I \neq J$), the constraint (2.28) can also be satisfied by a state vector $|B\rangle$ such that

$$(\hat{a}_i - z_0)^2 |B\rangle = 0, \quad \hat{a}_i |B\rangle \neq z_0 |B\rangle.$$

For a given set $\{z_1, z_2, \dots, z_k\}$, the solutions (2.34) span an N^k -dimensional vector space $W(\sigma_i)$ and the redefined baryons $\tilde{B}_{r_1 r_2 \dots r_k}$ are the coordinates of $W(\sigma_i)$. As stated in Eq. (2.27), $\tilde{B}_{r_1 r_2 \dots r_k}$ is in the direct product representation $\bigotimes_{i=1}^k \mathbf{N}$. They can be expressed in terms of the original baryons $B_{r_1 r_2 \dots r_k}^{n_1 n_2 \dots n_k}$ by using the relation

$$\tilde{B}_{r_1 r_2 \dots r_k} = \langle 0, r_1 ; \dots ; 0, r_k | e_1(\hat{a}_1) \dots e_k(\hat{a}_k) | B \rangle, \quad (2.36)$$

where $|0, r_1 ; \dots ; 0, r_k\rangle \equiv |0, r_1\rangle \otimes \dots \otimes |0, r_k\rangle$ are the ground states and e_I ($I = 1, \dots, k$) are the polynomials defined as

$$e_I(\lambda) \equiv \prod_{J \neq I} \frac{\lambda - z_J}{z_I - z_J}, \quad (e_I(z_J) = \delta_{IJ}). \quad (2.37)$$

Since this polynomial is ill-defined for coincident vortices $z_I = z_J$ (for $I \neq J$), the coherent state representation (2.34) is valid only for separated vortices. As we will see later, there exist well-defined coordinates of $W(\sigma_i)$ for arbitrary values of σ_i . They can be obtained from $\tilde{B}_{r_1 r_2 \dots r_k}$ by linear coordinate transformations with z_I -dependent coefficients. Hence the result that the linear space $W(\sigma_i)$ has the representation $\bigotimes_{i=1}^k \mathbf{N}$ holds for arbitrary values of σ_i , including the coincident cases ($z_I = z_J$), as well.

So far we have specified the state space $W(\sigma_i)$ to which the vortex state vectors belong. Now let us examine which vectors in $W(\sigma_i)$ can be actually allowed as vortex state vectors. The remaining constraint is the Plücker relation (2.26) which reads

$$\tilde{B}_{r_1 \dots r_i \dots r_k} \tilde{B}_{s_1 \dots s_i \dots s_k} = \tilde{B}_{r_1 \dots s_i \dots r_k} \tilde{B}_{s_1 \dots r_i \dots s_k}, \quad (2.38)$$

for each $i = 1, 2, \dots, k$. This is solved by

$$\tilde{B}_{r_1 r_2 \dots r_k} = \phi_{r_1}^1 \phi_{r_2}^2 \dots \phi_{r_k}^k, \quad (2.39)$$

Since the baryons are divided by $U(1)^{\mathbb{C}} \subset GL(k, \mathbb{C})$, the multiplication of a non-zero complex constant on each of $\vec{\phi}^I \in \mathbb{C}^N$ ($I = 1, \dots, k$) is unphysical. Therefore, each N -vector $\vec{\phi}^I = (\phi_1^I, \dots, \phi_N^I)$ parametrizes $\mathbb{C}P^{N-1}$.

We thus see that for separated vortices the baryon given in Eq. (2.34) can be written as an anti-symmetric product of “single vortex states”

$$|B\rangle = \hat{\mathcal{A}} \left[\left(\sum_{r_1=1}^N \phi_{r_1}^1 |z_1, r_1\rangle \right) \otimes \left(\sum_{r_2=1}^N \phi_{r_2}^2 |z_2, r_2\rangle \right) \otimes \dots \otimes \left(\sum_{r_k=1}^N \phi_{r_k}^k |z_k, r_k\rangle \right) \right]. \quad (2.40)$$

This means that the moduli space of the separated vortices is just a k -symmetric product of $\mathbb{C} \times \mathbb{C}P^{N-1}$ parametrized by the position of the vortices z_I and the orientation $\vec{\phi}^I$ [7]

$$\mathcal{M}^{k\text{-separated}} \simeq (\mathbb{C} \times \mathbb{C}P^{N-1})^k / \mathfrak{S}_k, \quad (2.41)$$

where \mathfrak{S}_k stands for the symmetric group. Note that the space of vortex states Eq. (2.40), which are just generic (anti-symmetrized) *factorized* states. It spans far fewer dimensions ($2Nk$) than might naïvely be expected for the product-states made of k vectors, which would have a dimension of the order of $2N^k$, ignoring the position moduli.

Remarks

As is clear – hopefully – from our construction, the use of the vortex “state vector” notion is here for convenience only, made for exhibiting the group-theoretic properties of the non-Abelian vortices. In other words we do not attribute to $|B\rangle$ any direct physical significance. Accordingly, we need not discuss the question of their normalization (metric on the vector space V) here. Note that two of the constraints (Eq. (2.22) and Eq. (2.25)) are indeed linear; the third, quadratic constraint (Eq. (2.26)) does not affect their normalization either.

It is tempting, on the other hand, to note that any choice of a metric in V would induce a metric on the vortex moduli space, which is of physical interest. As discussed briefly in Appendix B, however, a simple-minded choice of the metric for $|B\rangle$ does not lead to the fully correct behavior of the vortex interactions.

2.3 Highest-weight coincident vortices and $SU(N)$ irreducible orbits

Let us next consider k vortices on top of each other, all centered at the origin. Namely we focus our attention on the subspace of the moduli space specified by the condition

$$\sigma_i = 0 \quad \text{for all } i. \quad (2.42)$$

Since the coherent states of Eq. (2.30) are not the general solution to the constraint (2.28), the situation is now more complicated. To understand the structure of this subspace in detail, it is important to know how the $SU(N)_{C+F}$ acts on it. As we have seen, the moduli space of vortices can be described in terms of the vortex state vector endowed with a linear representation of the

$SU(N)$ action. We will denote the $SU(N)$ orbits of highest-weight vectors (to be defined below) the “*irreducible $SU(N)$ orbits*” since the vectors belong to irreducible representations on those orbits. In this subsection we classify irreducible $SU(N)$ orbits by Young tableaux.

The “highest-weight vectors” will be defined as the special configurations of ψ and Z satisfying the following conditions:

- Any $U(1)^{N-1}$ transformation in the Cartan subgroup of $SU(N)$ can be absorbed by a $GL(k, \mathbb{C})$ transformation. Namely, for an arbitrary diagonal matrix $D \in U(1)^{N-1}$, there exists an element $g \in GL(k, \mathbb{C})$ such that

$$\psi D = g \psi, \quad Z = g Z g^{-1}. \quad (2.43)$$

- Any infinitesimal $SU(N)$ transformation with a raising operator \hat{E}_α can be absorbed by an infinitesimal $SL(k, \mathbb{C})$ transformation. Namely, for an arbitrary lower triangular matrix L whose diagonal entries are all 1, there exists an element $\tilde{g} \in SL(k, \mathbb{C})$ such that

$$\psi L = \tilde{g} \psi, \quad Z = \tilde{g} Z \tilde{g}^{-1}. \quad (2.44)$$

Such configurations are classified by a non-increasing sequence of integers $\{l_1, l_2, \dots, l_{k_1}\}$ satisfying

$$N \geq l_1 \geq l_2 \geq \dots \geq l_{k_1} \geq 0, \quad l_1 + l_2 + \dots + l_{k_1} = k. \quad (2.45)$$

In other words, they are specified by Young tableaux (diagrams)⁹ with k boxes

$$\begin{array}{ccccc}
 & \overbrace{\hspace{1.5cm}}^{k_1} & & & \\
 \boxed{1} & \boxed{1} & \boxed{\cdots} & \boxed{\cdots} & \boxed{1} \\
 \boxed{2} & \boxed{2} & \boxed{\cdots} & \boxed{2} & & \\
 \boxed{\vdots} & \boxed{\vdots} & & & & \\
 \boxed{\vdots} & \boxed{l_2} & & & & \\
 \boxed{l_1} & & & & &
 \end{array} \quad (2.46)$$

where the height of the i -th column is l_i and the width of the i -th row is k_i . The total number of boxes is equal to the vortex winding number k . An example of a pair of matrices (ψ, Z) corresponding the highest-weight state is given in Fig. 1. For such a pair of matrices (ψ, Z) , one

⁹ In the following, the term “Young tableaux” is used to denote diagrams without numbers in the boxes (Young diagrams), unless otherwise stated.

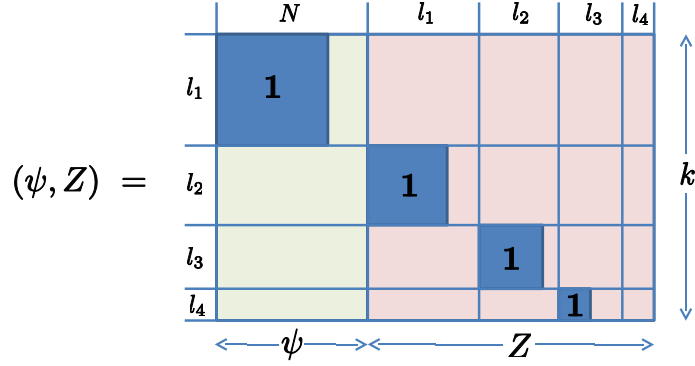


Fig. 1: An example of a k -by- $(N + k)$ matrix (ψ, Z) with $k_1 = 4$. The painted square boxes stand for unit matrices while the blank spaces imply that all their elements are zero.

can check the existence of g and \tilde{g} satisfying Eq. (2.43) and Eq. (2.44), given by

$$g = \begin{pmatrix} D_{l_1} & & \\ & \ddots & \\ & & D_{l_{k_1}} \end{pmatrix}, \quad \tilde{g} = \begin{pmatrix} L_{l_1} & & \\ & \ddots & \\ & & L_{l_{k_1}} \end{pmatrix}, \quad (2.47)$$

where k_1 is the number of boxes in the first row of the Young tableau, and D_{l_i} and L_{l_i} are the upper-left l_i -by- l_i minor matrices of D and L , respectively.¹⁰

The baryons corresponding to (ψ, Z) are given by

$$|B\rangle = \hat{\mathcal{A}}[|l_1\rangle \otimes |l_2\rangle \otimes \cdots \otimes |l_{k_1}\rangle], \quad |l_{n+1}\rangle \equiv |n, 1\rangle \otimes |n, 2\rangle \otimes \cdots \otimes |n, l_{n+1}\rangle. \quad (2.48)$$

We claim that this state is the highest-weight vector of the irreducible representation of $SU(N)$ specified by the Young tableau. This can be verified as follows. Since (ψ, Z) satisfy the condition Eq. (2.43), the baryons transform under the $U(1)^{N-1}$ transformation according to

$$|B\rangle \rightarrow \det g |B\rangle = \exp\left(i \sum_{i=1}^{l_1} k_i \theta_i\right) |B\rangle, \quad \sum_{i=1}^N \theta_i = 0, \quad (2.49)$$

where k_i is the number of boxes in the i -th row of the Young tableau. The weights of the $U(1)^{N-1}$ action can be read off in terms of k_i as

$$m_i = k_i - k_{i+1}, \quad (2.50)$$

where the integers $[m_1, m_2, \cdots, m_{N-1}]$ are the Dynkin labels. On the other hand, since (ψ, Z) satisfy the condition Eq. (2.44), the $SL(k, \mathbb{C})$ invariants $B_{r_1 r_2 \cdots r_k}^{n_1 n_2 \cdots n_k}$ are annihilated by the raising

¹⁰For example, $D_{l_i} = \text{diag}(e^{i\theta_1}, \cdots, e^{i\theta_{l_i}})$ for $D = \text{diag}(e^{i\theta_1}, \cdots, e^{i\theta_N})$.

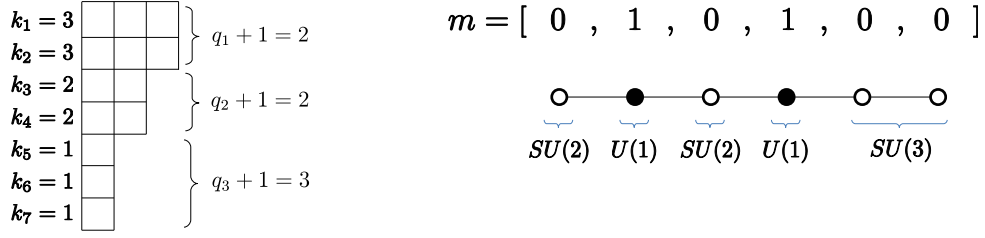


Fig. 2: An example with $N = 7$, $m = [0, 1, 0, 1, 0, 0]$ (and $k = 13$), $\mathcal{M}_{\text{orbit}} = \frac{SU(7)}{SU(2) \times SU(2) \times SU(3) \times U(1)^2}$. The black nodes in the Dynkin diagram denote the removed nodes [23].

operators

$$\hat{E}_\alpha |B\rangle = 0. \quad (2.51)$$

We have thus proved that (2.48) represents the highest-weight state of the representation (2.46) in the usual sense.

We define an “*irreducible $SU(N)$ orbit for the set of Dynkin labels: $[m_1, m_2, \dots, m_{N-1}]$ ” as an $SU(N)$ orbit of the corresponding highest-weight state. Note that this definition is obviously independent of the choice of $U(1)^{N-1} \in SU(N)$ in Eq. (2.43). It is known that such an orbit is a generalized flag manifold of the form $SU(N)/H$ with H being a subgroup of $SU(N)$ which acts on the highest-weight state as*

$$\hat{h} |B\rangle = e^{i\theta(\hat{h})} |B\rangle \sim |B\rangle, \quad \forall \hat{h} \in H. \quad (2.52)$$

The subgroup H can be specified by removing the nodes in the Dynkin diagram which correspond to non-zero Dynkin labels $m_i \neq 0$, i.e. it is specified by a painted Dynkin diagram [23]. Therefore, the irreducible orbits can be written as generalized flag manifolds¹¹

$$\mathcal{M}_{\text{orbit}} = \frac{SU(N)}{SU(q_1 + 1) \times \dots \times SU(q_{p+1} + 1) \times U(1)^p}, \quad (2.53)$$

where p ($1 \leq p \leq N - 1$) is the number of removed nodes and q_i ($i = 1, \dots, p + 1$) is the number of nodes in the connected component between the $(i - 1)$ -th and i -th removed nodes (see Fig. 2). The number p is denoted the rank of the Kähler coset space (2.53). One can also verify that an H -transformation on (ψ, Z) can indeed be absorbed by $GL(k, \mathbb{C})$ transformations.

It will now be shown that the irreducible orbits are the fixed-point set of the spatial rotation

$$(\psi, Z) \rightarrow (\psi, e^{i\theta} Z). \quad (2.54)$$

¹¹These orbits were studied in a non-systematic way in Ref. [24].

To see this, it is sufficient to check that the highest-weight state is invariant under the rotation (2.54), since the $SU(N)$ transformations commute with the spatial rotation. One way to show the invariance of the highest-weight state is to find a $GL(k, \mathbb{C})$ transformation which cancels the transformation (2.54) on the matrix of Fig. 1. A different, but easier, way is to check the invariance of the highest-weight state (2.45) under the action of the spatial rotations explicitly. Since the generator of the spatial rotation \hat{J} acts on the ground state $|0\rangle \equiv |0, r_1\rangle \otimes \cdots \otimes |0, r_k\rangle$ and the operators \hat{a}, \hat{a}^\dagger as (J is just a number operator)

$$\hat{J}|0\rangle = 0, \quad [\hat{J}, \hat{a}_i] = -\hat{a}_i, \quad [\hat{J}, \hat{a}_i^\dagger] = \hat{a}_i^\dagger, \quad (2.55)$$

the highest-weight state (2.45) is an eigenstate of \hat{J} , hence the state transforms as

$$|B\rangle \rightarrow \exp(i\theta\hat{J})|B\rangle = \exp\left(i\sum_{n=0}^{k_1-1} n l_{n+1}\theta\right)|B\rangle. \quad (2.56)$$

Since the phase of the state vector is unphysical, Eq. (2.56) shows that the highest-weight state is invariant under the spatial rotation. Therefore, the irreducible orbits are in the fixed-point set of the spatial rotation. The inverse also turns out to be true: we can show by using the moduli-matrix formalism that any fixed points of the spatial rotation are contained in one of the irreducible orbits. Therefore, the fixed-point set is precisely the disjoint union of the irreducible orbits.

All this can be seen more explicitly in terms of the original fields. The solution (H, A_μ) to the BPS equation (1.1) corresponding to the irreducible orbits can be determined from the fact that they are invariant under the spatial rotation

$$H(z, \bar{z}) \rightarrow H(e^{-i\theta}z, e^{i\theta}\bar{z}), \quad A_{\bar{z}}(z, \bar{z}) \rightarrow e^{i\theta}A_{\bar{z}}(e^{-i\theta}z, e^{i\theta}\bar{z}), \quad (2.57)$$

where $A_{\bar{z}} = A_1 + iA_2$. Let $(H^{(k)}, A_\mu^{(k)})$ be the solution of k ANO vortices situated at the origin $z = 0$. They transform under the rotation as

$$H^{(k)}(e^{-i\theta}z, e^{i\theta}\bar{z}) = e^{-ik\theta}H^{(k)}(z, \bar{z}), \quad A_{\bar{z}}^{(k)}(e^{-i\theta}z, e^{i\theta}\bar{z}) = A_{\bar{z}}^{(k)}(z, \bar{z}), \quad (2.58)$$

The solution on the irreducible orbits can be obtained by embedding the ANO solutions into diagonal components

$$H = U^\dagger \text{diag} \left(H^{(k_1)}, H^{(k_2)}, \dots, H^{(k_N)} \right) U, \quad A_{\bar{z}} = U^\dagger \left(A_{\bar{z}}^{(k_1)}, A_{\bar{z}}^{(k_2)}, \dots, A_{\bar{z}}^{(k_N)} \right) U, \quad (2.59)$$

where $U \in SU(N)_{C+F}$. Note that the sequence of the numbers $\{k_1, k_2, \dots, k_N\}$ can always be reordered as $k_1 \geq k_2 \geq \dots \geq k_N \geq 0$ by using the Weyl group $\mathfrak{S}_N \subset SU(N)_{C+F}$. This solution is invariant under the rotation since the phase factors of the Higgs fields can be absorbed by the following gauge transformation

$$H \rightarrow gH, \quad A_{\bar{z}} \rightarrow gA_{\bar{z}}g^\dagger, \quad g = U^\dagger \text{diag} (e^{-ik_1\theta}, e^{-ik_2\theta}, \dots, e^{-ik_N\theta}) U \in U(N)_C. \quad (2.60)$$

We can also see that the solution (2.59) is invariant under the same subgroup of $SU(N)$ as the state on the irreducible orbit specified by the Young tableau with k_i boxes in the i -th row. Therefore, the irreducible orbit with the set of Dynkin labels $[m_1, \dots, m_{N-1}]$ ($m_i = k_i - k_{i+1}$) corresponds to the BPS solutions of the form of Eq. (2.59).

In the next section, we will show that *a vortex state at a generic point on the moduli space is given by a linear superposition of vectors corresponding to various irreducible representations*. Furthermore, in Section 4, metrics for all irreducible $SU(N)$ orbits will be obtained by assuming that the metrics are Kähler and isometric under the $SU(N)$ action.

3 $SU(N)$ decomposition of general k vortex states

In this section we solve the constraints (2.25) and (2.26) in order to find the $SU(N)$ property of a general k -winding vortex. The cases of $k = 1, 2$ and 3 are solved concretely; a general recipe for the solution will be given, valid for any N and for any winding number k . A particular attention will be paid to the vortices with coincident centers. The results of these analyses provide the $SU(N)$ decomposition rule for a generic vortex state of a given winding number.

3.1 $k = 1$ vortices

$k = 1$ is a trivial example. In this case, we have

$$\sigma_1 = z_1, \quad |B\rangle = \sum_{r=1}^N \phi_r |z_1, r\rangle. \quad (3.1)$$

There is no nontrivial constraint, so that the moduli space is

$$\mathcal{M}^{k=1} = \mathbb{C} \times \mathbb{C}P^{N-1} \simeq \mathbb{C} \times \frac{SU(N)}{SU(N-1) \times U(1)}. \quad (3.2)$$

As $|B\rangle$ is in the fundamental representation of $SU(N)$, the orientational moduli space is given by the orbit of a vector in the fundamental representation.

3.2 Solution of the constraints for $k = 2$

This is the first case with nontrivial constraints.

$k = 2$ $U(N)$ vortices

With coordinates $\sigma_1 = z_1 + z_2 \in \mathbb{C}$ and $\sigma_2 = z_1 z_2 \in \mathbb{C}$, the linear constraints (2.25) in this case are given by

$$(\hat{a}_1 + \hat{a}_2) |B\rangle = \sigma_1 |B\rangle, \quad \hat{a}_1 \hat{a}_2 |B\rangle = \sigma_2 |B\rangle, \quad (3.3)$$

which are equivalent to the following equations for the baryonic invariants

$$B_{r \ s}^{n+1m} + B_{r \ s}^{nm+1} = \sigma_1 B_{r \ s}^{nm}, \quad B_{r \ s}^{n+1m+1} = \sigma_2 B_{r \ s}^{nm}. \quad (3.4)$$

In Section 2.2, we have seen that the solution can be expressed by the coherent states for separated vortices. Let us see what happens to the coherent states in the coincident limit. In the case of $k = 2$, the coherent state representation of the solution is given by

$$|B\rangle = \frac{1}{2} \tilde{B}_{r_1 r_2} \left(|z_1, r_1\rangle \otimes |z_2, r_2\rangle - |z_2, r_2\rangle \otimes |z_1, r_1\rangle \right). \quad (3.5)$$

It is convenient to decompose $\tilde{B}_{r_1 r_2}$ into the irreducible representations of $SU(N)$

$$\tilde{A}_{r_1 r_2} \equiv \frac{\tilde{B}_{r_1 r_2} - \tilde{B}_{r_2 r_1}}{2}, \quad \tilde{S}_{r_1 r_2} \equiv \frac{\tilde{B}_{r_1 r_2} + \tilde{B}_{r_2 r_1}}{2}. \quad (3.6)$$

Then, the solution can be rewritten as

$$|B\rangle = \left[\tilde{A}_{r_1 r_2} \cosh \frac{(z_1 - z_2)(\hat{a}_1^\dagger - \hat{a}_2^\dagger)}{2} + \tilde{S}_{r_1 r_2} \sinh \frac{(z_1 - z_2)(\hat{a}_1^\dagger - \hat{a}_2^\dagger)}{2} \right] \left| \frac{\sigma_1}{2}, r_1 \right\rangle \otimes \left| \frac{\sigma_1}{2}, r_2 \right\rangle,$$

where $\sigma_1 = z_1 + z_2$. If we naively take the coincident limit $z_2 \rightarrow z_1$, the symmetric part drops out

$$|B\rangle \rightarrow \tilde{A}_{r_1 r_2} \left| \frac{\sigma_1}{2}, r_1 \right\rangle \otimes \left| \frac{\sigma_1}{2}, r_2 \right\rangle. \quad (3.7)$$

Although this state satisfies the constraint (3.3), this is not the most general solution in the coincident case. To obtain the correct expression for the most general solution, let us redefine

$$A_{r_1 r_2} \equiv \tilde{A}_{r_1 r_2}, \quad S_{r_1 r_2} \equiv \frac{z_1 - z_2}{2} \tilde{S}_{r_1 r_2}. \quad (3.8)$$

Then, the solution (3.5) can be rewritten as

$$|B\rangle = \sum_{n=0}^{\infty} \frac{1}{(2n)!} w^n (\hat{a}_1^\dagger - \hat{a}_2^\dagger)^{2n} \left[A_{r_1 r_2} + \frac{1}{2n+1} S_{r_1 r_2} (\hat{a}_1^\dagger - \hat{a}_2^\dagger) \right] \left| \frac{\sigma_1}{2}, r_1 \right\rangle \otimes \left| \frac{\sigma_1}{2}, r_2 \right\rangle, \quad (3.9)$$

where we have introduced a square of the relative position as

$$w \equiv \frac{\sigma_1^2}{4} - \sigma_2 = \frac{(z_1 - z_2)^2}{4}. \quad (3.10)$$

In this expression, it is obvious that the symmetric part also survives in the coincident limit $w \rightarrow 0$

$$|B\rangle \rightarrow \left[A_{r_1 r_2} + S_{r_1 r_2} (\hat{a}_1^\dagger - \hat{a}_2^\dagger) \right] \left| \frac{\sigma_1}{2}, r_1 \right\rangle \otimes \left| \frac{\sigma_1}{2}, r_2 \right\rangle. \quad (3.11)$$

Therefore, Eq. (3.9) is the most general form of the solution which is valid also in the coincident limit. The symmetric and anti-symmetric tensors S_{rs} and A_{rs} in Eq. (3.9) are the well-defined coordinates of the vector space $W(\sigma_i)$ for arbitrary values of σ_i . Clearly these correspond to the decomposition of the tensor product $\mathbf{N} \otimes \mathbf{N}$ into irreducible representations of $SU(N)$. *A generic point on the moduli space is described by a superposition of the states belonging to different irreducible representations.*

In terms of A_{rs} and S_{rs} , the baryonic invariants can be read off from the solution using (2.20)

$$B_{rs}^{00} = A_{rs}, \quad B_{rs}^{10} = S_{rs} + \frac{\sigma_1}{2} A_{rs}, \quad B_{rs}^{11} = \sigma_2 A_{rs}, \quad \dots, \quad (3.12)$$

and hence, the Plücker conditions (2.26), which are the remaining constraints, can be rewritten as

$$A_{pq} A_{rs} + A_{pr} A_{sq} + A_{ps} A_{qr} = 0, \quad (3.13)$$

$$A_{pq} S_{rs} + A_{rp} S_{qs} + S_{ps} A_{qr} = 0, \quad (3.14)$$

$$w A_{pq} A_{rs} + S_{pr} S_{qs} - S_{ps} S_{qr} = 0. \quad (3.15)$$

By these constraints, the moduli space of two vortices is embedded into $\mathbb{C}^2 \times \mathbb{C}P^{N^2-1}$ which is parametrized by independent coordinates $\{\sigma_1, \sigma_2, A_{rs}, S_{rs}\}$.

Now, let us look into two different subspaces corresponding to the irreducible $SU(N)$ orbits. They are obtained by setting 1) $S_{rs} = 0$, $A_{rs} \neq 0$ and 2) $A_{rs} = 0$, $S_{rs} \neq 0$.

1) Consider first the subspace with $S_{rs} = 0$. Eq. (3.15) allows $S_{rs} = 0$ only in the coincident case $w = 0$. Note that Eq. (3.14) is automatically satisfied by $S_{rs} = 0$, and that Eq. (3.13) gives the ordinary Plücker conditions which embed the complex Grassmannian $Gr_{N,2}$ into a complex projective space $\mathbb{C}P^{N(N-1)/2-1} \simeq \{A_{pq}\}/\mathbb{C}^*$. We find therefore that the subspace $S_{rs} = 0$ is:

$$\mathcal{M}^{\mathbb{B}} \cong \mathbb{C} \times Gr_{2,N} \cong \mathbb{C} \times \frac{SU(N)}{SU(2) \times SU(N-2) \times U(1)}. \quad (3.16)$$

According to the results in the previous section, this is the irreducible $SU(N)$ orbit for \mathbb{B} .

2) In the other subspace characterized by $A_{rs} = 0$, we have a nontrivial constraint $S_{pr}S_{qs} = S_{ps}S_{qr}$. The general solution is

$$S_{rs} = \phi_r \phi_s, \quad \phi_r \in \mathbb{C}^N. \quad (3.17)$$

Here ϕ_r is nothing but the orientation vector given in Eq. (3.1), so $S_{rs} = \phi_r \phi_s$ corresponds to the $k = 2$ vortices with parallel orientations. The corresponding moduli subspace is given by

$$\mathcal{M}^{\square} \cong \mathbb{C}^2 \times \mathbb{C}P^{N-1} \cong \mathbb{C}^2 \times \frac{SU(N)}{SU(N-1) \times U(1)}, \quad (3.18)$$

which is indeed the other irreducible orbit, extended for generic w . We have thus identified the two moduli subspaces, the irreducible $SU(N)$ orbits of anti-symmetric and symmetric representations, respectively. They correspond to the vortex states in Eq. (3.9) without the second or the first term, respectively. The generic vortex state (3.9) is a linear superposition of these two states.

Note that in some cases the orbits of different representations are described by the same coset manifold. For example, both \square and $\square\square$ are given by $\mathbb{C}P^{N-1}$, see Eqs. (3.2) and (3.18). As we shall see in Section 4, however, the Kähler class completely specifies the representations and distinguishes the orbits belonging to different representations ¹².

More on $k = 2$ coincident $U(2)$ vortices

Let us study $k = 2$ vortices in the $U(2)$ case in some more detail by looking at another slice of the moduli space. This case in particular has been studied in the Refs. [13–18]. In this case, there exist only a singlet A_{12} and a triplet $\{S_{11}, S_{12}, S_{22}\}$ of $SU(2)$.

¹²Except for the cases of pairs of conjugate representations. They are found to be described by the same Kähler metric, i.e., by the same low-energy effective action. See Subsection 4.2 below.

Among the constraints (3.13)–(3.15), the only nontrivial one is

$$w(A_{12})^2 + S_{11}S_{22} - (S_{12})^2 = 0. \quad (3.19)$$

Let us consider the moduli space of coincident vortices which corresponds to the subspace $w = 0$. In this case, the above constraint is solved by $S_{rs} = \phi_r \phi_s$ again. Now, the moduli subspace is parametrized by the center of mass position $z_0 = \frac{\sigma_1}{2}$ and $\{\eta, \phi_1, \phi_2\}$ with $\eta \equiv A_{12}$. Thus, the vortex state is given, without constraints, by

$$|B\rangle_{w=0} = \eta |z_0\rangle_{\mathbf{1}} + \sum_{r,s=1}^2 \phi_r \phi_s |z_0; r, s\rangle_{\mathbf{3}}, \quad (3.20)$$

where the singlet $|z_0\rangle_{\mathbf{1}}$ and the triplet $|z_0; r, s\rangle_{\mathbf{3}}$ are given by

$$|z_0\rangle_{\mathbf{1}} \equiv |z_0, 1\rangle \otimes |z_0, 2\rangle - |z_0, 2\rangle \otimes |z_0, 1\rangle, \quad (3.21)$$

$$|z_0; r, s\rangle_{\mathbf{3}} \equiv (\hat{a}_1^\dagger - \hat{a}_2^\dagger) \left(|z_0, r\rangle \otimes |z_0, s\rangle + |z_0, s\rangle \otimes |z_0, r\rangle \right). \quad (3.22)$$

Note that the $\mathbb{C}^* \subset GL(k, \mathbb{C})$ acts as

$$\{\eta, \phi_1, \phi_2\} \sim \{\lambda^2 \eta, \lambda \phi_1, \lambda \phi_2\}, \quad \lambda \in \mathbb{C}^*. \quad (3.23)$$

Hence the moduli subspace for the two coincident vortices is found to be the two dimensional weighted projective space with the weights $(2, 1, 1)$

$$\mathcal{M}_{k=2}^{\text{coincident}} \cong \mathbb{C} \times W\mathbb{C}P_{(2,1,1)}^2 \cong \mathbb{C} \times \frac{\mathbb{C}P^2}{\mathbb{Z}_2}. \quad (3.24)$$

This is exactly the result obtained previously [14, 15]. Although this might be seen as just a reproduction of an old result, there is a somewhat new perspective on the irreducible representation of $SU(2)$. Here we would like to stress again that $A_{12} = \eta$ is the singlet while $S_{rs} = \phi_r \phi_s$ is the triplet. *Together* they form the coordinate of $W\mathbb{C}P_{(2,1,1)}^2$. In Fig. 3, we show the space $W\mathbb{C}P_{(2,1,1)}^2$ in the $|\phi_1|^2 - |\phi_2|^2$ plane with a natural metric given by $2|\eta|^2 + |\phi_1|^2 + |\phi_2|^2 = 1$. The states $\mathbf{3}$ and $\mathbf{1}$ live on the boundaries of $W\mathbb{C}P_{(2,1,1)}^2$; the points in the bulk of $W\mathbb{C}P_{(2,1,1)}^2$ are described by the superposition $\mathbf{1} \oplus \mathbf{3}$.

In Appendix C we discuss possible metrics on $W\mathbb{C}P_{(2,1,1)}^2$ and show that independently of the choice of the metric, they indeed yield at the diagonal edge of Fig. 3 the Fubini-Study metric with the same Kähler class on $\mathbb{C}P^1$.

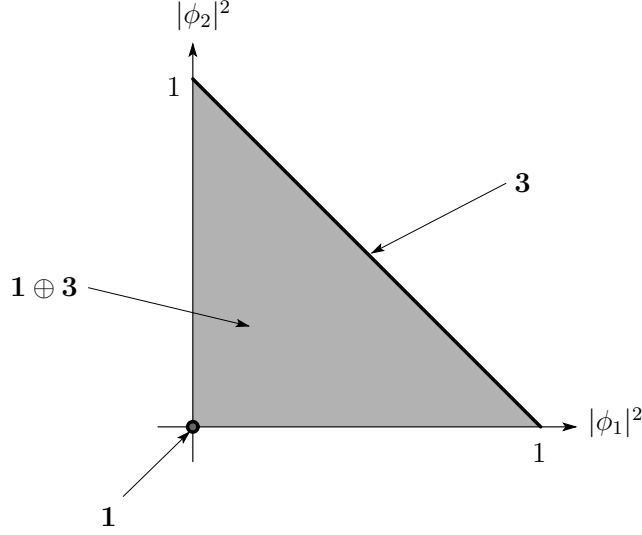


Fig. 3: $W\mathbb{C}P^2_{(2,1,1)}$ in the gauge $2|\eta|^2 + |\phi_1|^2 + |\phi_2|^2 = 1$. The diagonal edge corresponds to the triplet state **3** and the origin to the singlet state **1**. The bulk is a nontrivial superposition of **1** and **3**. The diagonal edge and the origin are the only irreducible orbits in this system.

3.3 Solution for the $k = 3$ coincident vortices

In this section, we consider $k = 3$ vortices sitting all at the origin, $\sigma_1 = \sigma_2 = \sigma_3 = 0$ ($z_1 = z_2 = z_3 = 0$). (The $k = 3$ vortex solutions of more general types – with generic center positions – will be discussed in Appendix D.) The constraint (2.25) reduces to

$$\hat{a}_1 \hat{a}_2 \hat{a}_3 |B\rangle = 0, \quad (\hat{a}_1 \hat{a}_2 + \hat{a}_2 \hat{a}_3 + \hat{a}_3 \hat{a}_1) |B\rangle = 0, \quad (\hat{a}_1 + \hat{a}_2 + \hat{a}_3) |B\rangle = 0, \quad (3.25)$$

which lead to $(\hat{a}_i)^3 |B\rangle = 0$ for $i = 1, 2, 3$. Taking into account the anti-symmetry condition (2.21), we obtain the following solution to the constraints (see Appendix D)

$$\begin{aligned} |B\rangle = & \left[A_{r_1 r_2 r_3} + \left(X_{r_1 r_2 r_3}^1 \hat{a}_1^\dagger + X_{r_1 r_2 r_3}^2 \hat{a}_2^\dagger + X_{r_1 r_2 r_3}^3 \hat{a}_3^\dagger \right) \right. \\ & - \frac{1}{2} \left(Y_{r_1 r_2 r_3}^1 (\hat{a}_2^\dagger - \hat{a}_3^\dagger)^2 + Y_{r_1 r_2 r_3}^2 (\hat{a}_1^\dagger - \hat{a}_3^\dagger)^2 + Y_{r_1 r_2 r_3}^3 (\hat{a}_1^\dagger - \hat{a}_2^\dagger)^2 \right) \\ & \left. - \frac{1}{2} S_{r_1 r_2 r_3} (\hat{a}_1^\dagger - \hat{a}_2^\dagger)(\hat{a}_2^\dagger - \hat{a}_3^\dagger)(\hat{a}_3^\dagger - \hat{a}_1^\dagger) \right] |0, r_1\rangle \otimes |0, r_2\rangle \otimes |0, r_3\rangle, \end{aligned} \quad (3.26)$$

where $Y_{r_1 r_2 r_3}^i$ ($i = 1, 2, 3$) and $X_{r_1 r_2 r_3}^i$ ($i = 1, 2, 3$) are tensors satisfying

$$Y_{r_1 r_2 r_3}^1 + Y_{r_1 r_2 r_3}^2 + Y_{r_1 r_2 r_3}^3 = 0, \quad X_{r_1 r_2 r_3}^1 + X_{r_1 r_2 r_3}^2 + X_{r_1 r_2 r_3}^3 = 0. \quad (3.27)$$

The tensors S, Y, X, A have the following index structures

$$S_{r_1 r_2 r_3} = S_{r_{\rho(1)} r_{\rho(2)} r_{\rho(3)}}, \quad (3.28)$$

$$Y_{r_1 r_2 r_3}^i = \text{sign}(\rho) Y_{r_{\rho(1)} r_{\rho(2)} r_{\rho(3)}}^{\rho(i)}, \quad (3.29)$$

$$X_{r_1 r_2 r_3}^i = \text{sign}(\rho) X_{r_{\rho(1)} r_{\rho(2)} r_{\rho(3)}}^{\rho(i)}, \quad (3.30)$$

$$A_{r_1 r_2 r_3} = \text{sign}(\rho) A_{r_{\rho(1)} r_{\rho(2)} r_{\rho(3)}}, \quad (3.31)$$

where ρ denotes elements of the symmetric group \mathfrak{S}_3 . The first and last equation show that $S_{r_1 r_2 r_3}$ and $A_{r_1 r_2 r_3}$ are totally symmetric and anti-symmetric, respectively. The second (third) equation indicates that only one of Y^1, Y^2, Y^3 (X^1, X^2, X^3) is independent. Hence we arrive at a natural correspondence between the baryons and the Young tableaux as

$$A_{r_1 r_2 r_3} : \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}, \quad X_{r_1 r_2 r_3}^i : \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \quad Y_{r_1 r_2 r_3}^i : \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \quad S_{r_1 r_2 r_3} : \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}. \quad (3.32)$$

This looks perfectly consistent with the standard decomposition of $\square \otimes \square \otimes \square$.

Actually this is not quite straightforward, and this example nicely illustrates the subtlety alluded in the Introduction. As we have seen in the previous section, there is a one-to-one correspondence between the highest-weight states of the baryons $|B\rangle$ and the Young tableaux with k boxes of a definite type. This means that there is only one vortex state of highest weight, corresponding to the mixed-symmetry Young tableau¹³. However, we seem to have Y and X in (3.32), both of which correspond to the same Young tableau. This apparent puzzle is solved by looking at the following Plücker relation rewritten in terms of S, X, Y, A

$$(Y_{rst}^1)^2 = -S_{rsr} X_{tst}^1 - X_{srs}^1 S_{rtt} + X_{srt}^1 S_{rts}, \quad (\text{no sum over } r, s, t), \quad (3.33)$$

which shows that the tensor Y is determined in terms of the others up to a sign. This implies that no solution to Eq. (3.33) of “pure Y ” type, i.e., with $Y \neq 0, A = S = X = 0$, exists. Hence we have verified the one-to-one correspondence between the highest-weight baryon states $|B\rangle$ and the Young tableaux, as in Figure 4.

By setting two among S, X or A to be zero, we obtain the corresponding $SU(N)$ irreducible

¹³In contrast to the standard composition-decomposition rule for three distinguishable objects in the \mathbf{N} representation, *two* inequivalent highest weight states in the same irreducible representation, described by the same mixed-type Young tableau, will appear. This is not so for our k vortices.

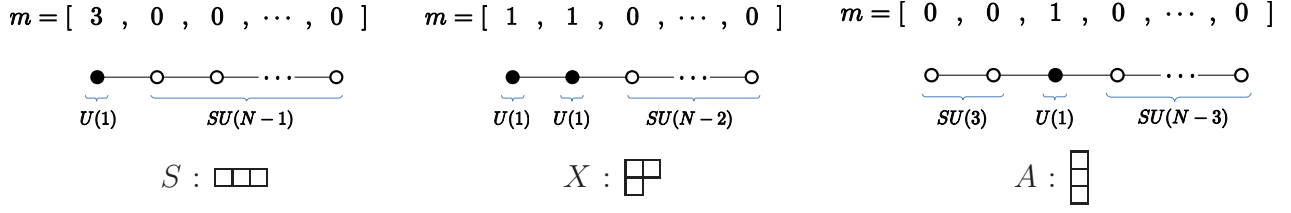


Fig. 4: The irreducible orbits in the moduli space of $k = 3$ vortices.

orbits, which can be immediately read off from the Young tableaux as (for $N \geq k = 3$)

$$\mathcal{M}^S \cong \frac{SU(N)}{SU(N-1) \times U(1)} \cong \mathbb{C}P^{N-1}, \quad (3.34)$$

$$\mathcal{M}^X \cong \frac{SU(N)}{SU(N-2) \times U(1)^2}, \quad (3.35)$$

$$\mathcal{M}^A \cong \frac{SU(N)}{SU(3) \times SU(N-3) \times U(1)} \cong Gr_{N,3}. \quad (3.36)$$

Due to the existence of Y , the whole subspace with $\sigma_i = 0$ is more complicated than the $k = 2$ case. The simplest nontrivial case $N = 2$ ($SU(2)$ global symmetry) somewhat enlightens our understanding. In that case, A is identically zero and the following parametrization using the coordinates $\{\eta, \xi^1, \xi^2, \phi_1, \phi_2\} \in \mathbb{C}^5$

$$X_{r12}^1 = \epsilon_{rs} \xi^s, \quad Y_{r12}^1 = \eta \phi_r, \quad S_{rst} = \phi_r \phi_s \phi_t, \quad r, s, t = 1, 2 \quad (3.37)$$

solves all of the Plücker relations except for

$$\eta^2 = \xi^r \phi_r. \quad (3.38)$$

Therefore, η is a locally dependent coordinate. Since the equivalence relation is

$$\{\xi^r, \eta, \phi_r\} \simeq \{\lambda^3 \xi^r, \lambda^2 \eta, \lambda \phi_r\}, \quad (3.39)$$

the moduli space in this case is a hypersurface in $W\mathbb{C}P_{(3,3,2,1,1)}^4 \simeq \mathbb{C}P^4/\mathbb{Z}_3$. The irreducible orbits corresponding to S and X are the subspaces obtained by setting $\xi^r = 0$ or $\phi_r = 0$, respectively.

Both of them are isomorphic to

$$\mathcal{M}^S \cong \mathcal{M}^X \cong \frac{SU(2)}{U(1)} \cong \mathbb{C}P^1. \quad (3.40)$$

According to the results of the next section, however, they are characterized by the different Kähler classes while their Kähler potentials are given by

$$K \simeq \begin{cases} 3r \log |\phi_r|^2 & \text{as } |\xi_i|^2 \rightarrow 0 \\ r \log |\xi^r|^2 & \text{as } |\phi_i|^2 \rightarrow 0 \end{cases}. \quad (3.41)$$

3.4 Generalization to arbitrary winding number

In this section, we comment on a generalization to the case of an arbitrary winding number k . As we have seen in the $k = 2, 3$ cases, the coherent states (2.30) become insufficient to describe the general solution to the constraint (2.25) when two or more vortex centers coincide. The procedure to obtain the general solution for $k = 3$ vortices can be generalized to the case of arbitrary k as follows. Let $|S; r_1, \dots, r_k; \{z_i\}\rangle$ be the following linear combination of the coherent states

$$|S; r_1, \dots, r_k; \{z_i\}\rangle \equiv \frac{1}{k! \Delta} \sum_{\rho \in \mathfrak{S}_k} \text{sign}(\rho) \hat{\rho} \hat{v} \hat{\rho}^{-1} |0, r_1\rangle \otimes \dots \otimes |0, r_k\rangle, \quad (3.42)$$

where the polynomial Δ and the operators \hat{v} are defined by

$$\Delta \equiv \prod_{I > J} (z_I - z_J), \quad \hat{v} \equiv \exp \left(\sum_{i=1}^k z_i a_i^\dagger \right); \quad (3.43)$$

$\hat{\rho} \hat{v} \hat{\rho}^{-1}$ then reads

$$\hat{\rho} \hat{v} \hat{\rho}^{-1} = \exp \left(z_1 \hat{a}_{\rho^{-1}(1)}^\dagger + z_2 \hat{a}_{\rho^{-1}(2)}^\dagger + \dots + z_k \hat{a}_{\rho^{-1}(k)}^\dagger \right). \quad (3.44)$$

This state vector (3.42) is a solution of the constraint (2.25) which is well-defined even in the coincident limit $z_I \rightarrow z_J$:

$$|S; r_1, \dots, r_k; \{z_i\}\rangle \rightarrow \Delta(\hat{a}_1^\dagger, \dots, \hat{a}_k^\dagger) |0, r_1\rangle \otimes \dots \otimes |0, r_k\rangle. \quad (3.45)$$

Other well-defined solutions can be obtained by acting with polynomials of annihilation operators \hat{a}_i on $|S; r_1, \dots, r_k; \{z_i\}\rangle$. The linearly independent solutions are generated by the polynomials $h_i(\hat{a}_1, \dots, \hat{a}_k)$ satisfying the following property¹⁴ for arbitrary symmetric polynomials P :

$$\langle 0 | h_i(\hat{a}_1, \dots, \hat{a}_k) P(\hat{a}_1^\dagger, \dots, \hat{a}_k^\dagger) = 0, \quad (3.46)$$

where $\langle 0 | \equiv \langle 0, r_1 | \otimes \dots \otimes \langle 0, r_k |$. Such polynomials $h_i(\hat{a}_1, \dots, \hat{a}_k)$ span a $k!$ -dimensional vector space H on which the symmetric group \mathfrak{S}_k acts linearly¹⁵

$$\hat{\rho} h_i(\hat{a}_1, \dots, \hat{a}_k) \hat{\rho}^{-1} = h_i(\hat{a}_{\rho^{-1}(1)}, \dots, \hat{a}_{\rho^{-1}(k)}) = g_i^j(\rho) h_j(\hat{a}_1, \dots, \hat{a}_k), \quad (3.47)$$

¹⁴The conditions (3.46) can be written in an alternative, equivalent form $P(\partial_1, \dots, \partial_k) h_i(\eta_1, \dots, \eta_k) = 0$, where $\partial_i \equiv \partial / \partial \eta_i$.

¹⁵The representation of H is isomorphic to the regular representation of \mathfrak{S}_k .

where $g_i^j(\rho)$ is a matrix corresponding to the transformation $\rho \in \mathfrak{S}_k$. By using a linearly independent basis $\{h_i\}$, the general solution to Eq. (2.25) can be written as a superposition of $h_i |S; r_1 \cdots r_k\rangle$

$$|B\rangle = \sum_{r_1, \dots, r_k} \sum_{i=1}^{k!} X_{r_1 \dots r_k}^i h_i(\hat{a}_1, \dots, \hat{a}_k) |S; r_1, \dots, r_k; \{z_i\}\rangle. \quad (3.48)$$

Since $|S; r_1, \dots, r_k; \{z_i\}\rangle$ is well-defined for arbitrary vortex positions, this expression of the general solution is valid even in the coincident limit. Taking into account the constraint Eq. (2.22), we find that $X_{r_1 \dots r_k}^i$ should have the following index structure

$$X_{r_1 \dots r_k}^i = X_{r_{\rho^{-1}(1)} \dots r_{\rho^{-1}(k)}}^i g_i^j(\rho), \quad \text{for all } \rho \in \mathfrak{S}_k. \quad (3.49)$$

This condition reduces the number of degrees of freedom to $N^k = \dim W(\sigma_i)$. Since Eq. (2.30) and Eq. (3.48) are related by the change of basis from coherent states to $h_i |S; r_1, \dots, r_k; \{z_i\}\rangle$, the coordinates $X_{r_1 \dots r_k}^i$ can be obtained from $\tilde{B}_{r_1 \dots r_k}$ by a linear coordinate transformation with z_I -dependent coefficients. Therefore, it is obvious that $X_{r_1 \dots r_k}^i$ transforms under $SU(N)$ as a multiplet in the direct product representation $\otimes_{i=1}^k \mathbf{N}$. We can also confirm this fact by decomposing the $k!$ -dimensional vector space H into the irreducible representations of the symmetric group \mathfrak{S}_k . They are classified by the standard Young tableaux with k boxes (Young tableaux with increasing numbers in each row and column) and correspondingly, the set of the coefficients $\{X_{r_1 \dots r_k}^i\}$ can also be decomposed into subsets classified by the standard Young tableaux. Eq. (3.49) then tells us that the subset of $X_{r_1 \dots r_k}^i$ for each irreducible representation of \mathfrak{S}_k forms a multiplet in the irreducible representation of $SU(N)$ specified by the corresponding Young tableau.

Finally, the remaining constraint (2.26) can be rewritten by using the relation (2.20) to quadratic constraints for $X_{r_1 \dots r_k}^i$, which give the vortex moduli space as a subspace in $\mathbb{C}^k \times \mathbb{C}P^{N^k}$.

4 Kähler potential on irreducible $SU(N)$ orbits

In this section we will obtain the metric on each of the irreducible orbits inside the vortex moduli space \mathcal{M}_k by use of a symmetry argument. We only use the fact that the metric of the whole vortex moduli space is Kähler and has an $SU(N)$ isometry.

One of the most important characteristics of non-Abelian vortices is that they possess internal orientational moduli. These arise when the vortex configuration breaks the $SU(N)_{\text{C+F}}$ symmetry

to its subgroup $H \subset SU(N)$. For a single vortex, it is broken to $SU(N-1) \times U(1)$ and the moduli space is homogeneous. On the other hand, the moduli space for multiple vortices, i.e. $k > 1$, is not homogeneous and has some anisotropic directions (even if we restrict ourselves to consider the subspace of coincident vortices). Consequently, the shape of the metric at generic points cannot be determined from the symmetry alone. The metric is not isometric along such a direction, and the isotropic subgroup H (and the orbit $SU(N)/H$) can change as we move along such a direction in \mathcal{M}_k .¹⁶ The moduli space \mathcal{M}_k contains all irreducible $SU(N)$ orbits associated with all possible Young tableaux having k boxes, as its subspaces which are invariant under the action of the spatial rotation. In the following, we uniquely determine the metrics for all irreducible $SU(N)$ orbits.

The irreducible orbits are all Kähler manifolds although generic $SU(N)$ orbits are not.¹⁷ We shall derive the Kähler potentials instead of the metrics directly.

The pair of matrices (ψ, Z) corresponding to generic points on an orbit is obtained by acting with $SU(N)$ on a specific configuration (ψ_0, Z_0) . Let us decompose any element $\mathcal{U} \in SU(N)$ as

$$\mathcal{U} = LDU, \quad (4.1)$$

where D is a diagonal matrix of determinant one and L (U) is a lower (upper) triangular matrix whose diagonal elements are all 1. This is called the LDU decomposition.¹⁸ In this case, the matrix \mathcal{U} is a unitary matrix $\mathcal{U}\mathcal{U}^\dagger = \mathbf{1}$, and hence the matrices L , D and U are related by

$$UU^\dagger = (LD)^{-1}(LD)^{\dagger-1}. \quad (4.2)$$

Therefore, once the matrix U is given, the lower triangular matrix LD is uniquely determined up to multiplication of diagonal unitary matrices u as $LD \rightarrow uLD$. That is, entries of U are complex coordinates of the flag manifold $SU(N)/U(1)^{N-1}$.

¹⁶This usually occurs in supersymmetric theories with spontaneously broken global symmetries and is called the supersymmetric vacuum alignment [25]. This phenomenon was discussed for non-Abelian vortices in Ref. [24] and for domain walls in Ref. [26]. For non-Abelian SO , USp vortices see Ref. [27].

¹⁷All irreducible $SU(N)$ orbits, which are the set of zeros of the holomorphic Killing vector for the spatial rotation, can be obtained as subspaces in \mathcal{M}_k by imposing certain *holomorphic* conditions. The latter takes the form (apart from the co-axial condition $\sigma_i = 0$) $B = 0$ for baryons which are not in a pure irreducible representation. Therefore the Kähler metrics are induced by these constraints from the Kähler metric on \mathcal{M}_k . It is an interesting question if a Kählerian coset space in \mathcal{M}_k always corresponds to an irreducible orbit.

¹⁸An invertible matrix admits an LDU decomposition if and only if all its principal minors are non-zero.

Let ψ_0 and Z_0 be matrices of the form given in Fig. 1 and $m = [m_1, m_2, \dots, m_{N-1}]$ be the set of Dynkin labels of the corresponding highest-weight state. Since the matrices ψ_0 and Z_0 satisfy the conditions (2.43) and (2.44), LD can be always absorbed by $g \in GL(k, \mathbb{C})$ and $\tilde{g} \in SL(k, \mathbb{C})$ given in Eq. (2.47)

$$\psi_0 \mathcal{U} = (\tilde{g}g) \psi_0 U, \quad Z_0 = (\tilde{g}g) Z_0 (\tilde{g}g)^{-1}. \quad (4.3)$$

This implies that a pair (ψ, Z) parametrizing the irreducible $SU(N)$ orbit is given by

$$\psi_{\text{orbit}} = \psi_0 U, \quad Z_{\text{orbit}} = Z_0, \quad U = \begin{pmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1,N} \\ & 1 & u_{23} & \cdots & u_{2,N} \\ & & 1 & \ddots & \vdots \\ & & & \ddots & u_{N-1,N} \\ & & & & 1 \end{pmatrix}, \quad u_{ij} \in \mathbb{C}. \quad (4.4)$$

The vortex state constructed by the latter is obtained as

$$|B_{\text{orbit}}\rangle \equiv |B(\psi_{\text{orbit}}, Z_{\text{orbit}})\rangle = \hat{U} |B(\psi_0, Z_0)\rangle = \det g^{-1} \hat{\mathcal{U}} |B(\psi_0, Z_0)\rangle, \quad (4.5)$$

with operators \hat{U} and $\hat{\mathcal{U}}$ corresponding to U and \mathcal{U} respectively.

In supersymmetric theories, ψ and Z can be regarded as chiral superfields. The complex parameters contained in U are also lifted to chiral superfields and can be regarded as Nambu-Goldstone zero-modes of $SU(N)/U(1)^{N-1}$.¹⁹ If $m_i \neq 0$ for all $i = 1, \dots, N-1$, then $SU(N)$ is broken to the maximal Abelian subgroup (the maximal torus) $U(1)^{N-1}$ and all the parameters u_{ij} are physical zero modes. One can easily check that the dimension of the flag manifold $SU(N)/U(1)^{N-1}$ counts the degrees of freedom in U . On the other hand, if $m_i = 0$ for some i 's, then the unbroken group H is enlarged from the maximal torus $U(1)^{N-1}$ to $SU(N)/H$ being generalized flag manifolds, from which we can further eliminate some of u_{ij} by using $GL(k, \mathbb{C})$.

Since the vortex moduli space \mathcal{M}_k has an $SU(N)$ isometry, the Kähler potential for \mathcal{M}_k , which is a real function of σ_i and B , should be invariant under the $SU(N)$ transformation

$$K(|B\rangle) = K(\hat{\mathcal{U}}|B\rangle), \quad (4.6)$$

¹⁹The generic Kähler potential on $SU(N)/U(1)^{N-1}$, which contains $N-1$ free parameters (Kähler classes), can be obtained from the method of supersymmetric non-linear realizations [28]. When all chiral superfields contain two Nambu-Goldstone scalars as in our case, they are called the pure realizations.

where $|B\rangle$ is the vortex state vector satisfying all the constraints (2.22), (2.25) and (2.26). Furthermore, the \mathbb{C}^* transformations on the Kähler potential should be absorbed by the Kähler transformations

$$K(e^\lambda |B\rangle) = K(|B\rangle) + f(\lambda) + \overline{f(\lambda)}, \quad (4.7)$$

since the \mathbb{C}^* action on $|B\rangle$ gives a physically equivalent state $e^\lambda |B\rangle \sim |B\rangle$. Note that this transformation can be absorbed only when λ is holomorphic in the moduli parameters. We can easily show that the function $f(\lambda)$ has the following properties

$$f(2\pi i) + \overline{f(2\pi i)} = f(0) + \overline{f(0)}, \quad (4.8)$$

$$f(\lambda_1 + \lambda_2) + \overline{f(\lambda_1 + \lambda_2)} = f(\lambda_1) + \overline{f(\lambda_1)} + f(\lambda_2) + \overline{f(\lambda_2)}. \quad (4.9)$$

From these relations the form of the function f can be determined as

$$f(\lambda) + \overline{f(\lambda)} = r(\lambda + \bar{\lambda}), \quad r \in \mathbb{R}. \quad (4.10)$$

Now we are ready to derive the Kähler potentials for the irreducible $SU(N)$ orbits. With the above assumptions, the Kähler potential for the $SU(N)$ orbit can be calculated as

$$\begin{aligned} K(u_{ij}, \bar{u}_{ij}) &\equiv K(|B_{\text{orbit}}\rangle) = K(\det g^{-1} \hat{\mathcal{U}} |B_0\rangle) \\ &= K(|B_0\rangle) - r \log |\det g|^2, \end{aligned} \quad (4.11)$$

where $B_0 = B(\psi_0, Z_0)$. Since the first term of Eq. (4.11) is a constant, it can be eliminated by a Kähler transformation. It follows from Eqs. (4.2) and (2.47) that

$$K(u_{ij}, \bar{u}_{ij}) = -r \log |\det g|^2 = r \sum_{l=1}^{N-1} m_l \log \det(U_l U_l^\dagger), \quad (4.12)$$

where U_l are l -by- N minor matrices of U given by

$$U_l = \begin{pmatrix} 1 & u_{12} & \cdots & u_{1,l} & u_{1,l+1} & \cdots & u_{1,N} \\ & 1 & \ddots & \vdots & \vdots & & \vdots \\ & & \ddots & u_{l-1,l} & \vdots & & \vdots \\ & & & 1 & u_{l,l+1} & \cdots & u_{l,N} \end{pmatrix}. \quad (4.13)$$

Note that if $m_l = 0$ for some l 's, the dimension of the manifold decreases in a way that is consistent with the enhancement of the symmetry H .

The coefficients $r m_l$ of the terms in the Kähler potential (4.12) determine the Kähler class of the manifold. As noted in the footnote 19 the generic Kähler potential contains $N - 1$ free parameters, which is now determined from the set of Dynkin labels $[m_1, m_2, \dots, m_{N-1}]$. We see that the Kähler classes are quantized in integers multiplied by r which implies that these Kähler manifolds are Hodge. This can be expected from the Kodaira theorem stating that Hodge manifolds are all algebraic varieties, i.e. they can be embedded into some projective space $\mathbb{C}P^n$ by holomorphic constraints.

The overall constant r of the Kähler potential cannot be determined by the above argument based on symmetry. It can however be obtained by a concrete computation, for instance, $k = 1$ vortex ($m = [1, 0, \dots, 0]$) results in Refs. [3–6]

$$r = \frac{4\pi}{g^2}, \quad (4.14)$$

which matches the result (2.5) based on the D -brane picture [1]. It can be also determined from the charge of instantons trapped inside a vortex [5].

Recently, some of us constructed [29] the world-sheet action and computed the metrics explicitly from first principles for the vortices in SO , USp and SU theories, generalizing the work of Refs. [4, 6]. The systems considered include the cases of some higher-winding vortices in $U(N)$ and $SO(2N)$ theories: the results found there are in accordance with the general discussion given here.

4.1 Examples

In this subsection we provide two examples with $N = 2$ and $N = 3$ to illustrate the determination of the Kähler potentials.

4.1.1 $N = 2$

To be concrete, let us take some simple examples for $N = 2$. For simplicity, we first consider the $k = 2$ case. There are two highest-weight states: the triplet and singlet, for which ψ_0 and Z_0 take the form, see Fig. 1,

$$(\psi, Z)_{\blacksquare} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right), \quad (\psi, Z)_{\blacksquare} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right). \quad (4.15)$$

In the former case $SU(2)$ is broken to $U(1)$ and the orbit is $SU(2)/U(1) \cong \mathbb{CP}^1$. Applying Eq. (4.12), we obtain the Kähler potential for the Fubini-Study metric on \mathbb{CP}^1

$$K_{N=2} = 2r \log(1 + |a|^2), \quad U = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}. \quad (4.16)$$

On the other hand, $SU(2)$ is unbroken in the singlet case. Indeed ψ_0 is just the unit matrix, so that an arbitrary $SU(2)$ transformation can indeed be canceled by $GL(2, \mathbb{C})$.

This can be easily extended to the generic case with $k > 2$. In the case of $k_1 > k_2$, $SU(2)$ is broken to $U(1)$ while if $k_1 = k_2$, $SU(2)$ is unbroken. From Eq. (4.12), we find the Kähler potential for the Fubini-Study metric on \mathbb{CP}^1 for $k_1 > k_2$:

$$K_{N=2} = r m_1 \log(1 + |a|^2), \quad m_1 = k_1 - k_2, \quad (4.17)$$

while the orbits are always \mathbb{CP}^1 for arbitrary k_1 and k_2 ($k_1 > k_2$), one can distinguish them by looking at the Kähler class $rm_1 = r(k_1 - k_2)$. For instance, one can distinguish two \mathbb{CP}^1 's in Eqs. (3.2) and (3.18) for one and two vortices, respectively.

4.1.2 $N = 3$

Next, let us study the $N = 3$ case. There are four different types according to the Young tableaux and the unbroken groups H , see Table 1. We parametrize the matrix U as

$$U = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}. \quad (4.18)$$

The complex parameters a, b, c are (would-be) Nambu-Goldstone zero-modes associated with $SU(3) \rightarrow H$. Applying Eq. (4.12), we find

$$K_{N=3} = r m_1 \log(1 + |a|^2 + |b|^2) + r m_2 \log(1 + |c|^2 + |b - ac|^2), \quad (4.19)$$

with $m_1 = k_1 - k_2$ and $m_2 = k_2 - k_3$. When $m_1 > 0$ and $m_2 > 0$ ($k_1 > k_2 > k_3$), this represents the Kähler potential for the Kähler manifold $SU(3)/U(1)^2$ with a particular choice of the complex structure [30]. When $m_1 > 0$ and $m_2 = 0$ ($k_1 > k_2 = k_3$), the parameter c disappears from the Kähler potential and hence it reduces to

$$K_{m_2=0} = r m_1 \log(1 + |a|^2 + |b|^2), \quad (4.20)$$





	$k_1 > k_2 > k_3$	$k_1 > k_2 = k_3$	$k_1 = k_2 > k_3$	$k_1 = k_2 = k_3$
YT				
H	$U(1)^2$	$U(1) \times SU(2)$	$U(1) \times SU(2)$	$SU(3)$

Table 1: Four different types of $N = 3$ coincident vortices.

which is nothing but the Kähler potential of $\mathbb{CP}^2 \simeq SU(3)/[U(1) \times SU(2)]$. When $m_1 = m_2 = 0$ ($k_1 = k_2 = k_3$), $SU(3)$ is unbroken, so that the orbit is just a point (with a vanishing Kähler potential).

4.2 Conjugate orbits

Note that in the $SU(3)$ example discussed in the last subsection the replacement

$$a \rightarrow -c, \quad b \rightarrow ac - b, \quad c \rightarrow -a \quad (4.21)$$

together with the exchange $m_1 \leftrightarrow m_2$, leaves invariant the Kähler potential (4.19). In other words, irreducible orbits for $m = [m_1, m_2]$ and $m = [m_2, m_1]$ are identical. In fact, this is a special case of duality between two $SU(N)$ conjugate representations, relating the irreducible orbits for $[m_1, m_2, \dots, m_{N-1}]$ to the one with $[m_{N-1}, m_{N-2}, \dots, m_1]$. As we are interested here in the motion of the orientational moduli parameters only, it is very reasonable that we find the same Kähler metric for a vortex in \mathbf{r} representation and another in \mathbf{r}^* representation.

Generalization to arbitrary (N, k) of the mapping (4.21) leaving the Kähler potential invariant is given by

$$\begin{aligned} [m_1, m_2, \dots, m_{N-1}] &\leftrightarrow [m_{N-1}, m_{N-2}, \dots, m_1], \\ U &\leftrightarrow E (U^T)^{-1} E, \end{aligned} \quad (4.22)$$

where $(E)_{ij} = \delta_{i, N-j+1}$.

Coming back to the concrete $SU(3)$ examples in Subsection 4.1.2, the case with $(k_1, k_2, k_3) = (2, 1, 0)$ corresponds to $\mathbf{8}$ of $SU(3)$ which of course is self-dual. A pair of $(k_1, k_2, k_3) = (3, 3, 0)$ and $(4, 1, 1)$ provides a nontrivial example of duality between two different irreducible orbits: they correspond to $\mathbf{10}^*$ and $\mathbf{10}$, respectively. Finally, the orbits $(k_1, k_2, k_3) = (5, 4, 0)$ and $(k_1, k_2, k_3) = (6, 2, 1)$ belong to the pair of irreducible representations, $\mathbf{35}^*$ and $\mathbf{35}$.

Actually, these examples are special, in the sense that the pairs have the same winding number. This is not necessary. The equality of the Kähler potential (the same effective action) for a pair of conjugate orbits defined above, holds for pairs of vortices of unequal winding numbers as well, as the above proof does not depend on the winding number, but on the Dynkin labels only. For instance, the $k = 1$ vortex in $SU(N)$, $m = [1, 0, \dots, 0]$ (belonging to \mathbf{N}), has the same Kähler potential as the totally antisymmetric vortex of winding number $k = N - 1$, $m = [0, \dots, 0, 1]$. The latter transforms as \mathbf{N}^* .

When the condition $\frac{2k}{N} \in \mathbb{Z}$ is met, it is possible to have pairs of conjugate vortices with the same k (the same tension) and belonging to conjugate representations, as in the concrete $SU(3)$ examples above.

5 Summary and outlook

By using the Kähler-quotient construction we have investigated the moduli spaces of higher-winding BPS non-Abelian vortices in $U(N)$ theory, for the purpose of clarifying the transformation properties of the points in the moduli under the exact global $SU(N)$ symmetry group. In the case of vortices with distinct centers, the moduli space is basically just the symmetrized direct product of those of individual vortices, $(\mathbb{C} \times \mathbb{C}P^{N-1})^k / \mathfrak{S}_k$. It turns out to be a rather nontrivial problem to exhibit the group-theoretic properties of the points in the submoduli, corresponding to the vortex solutions with a common center. The results found show that they do behave as a superposition of various “vortex states” corresponding to the irreducible representations, appearing in the standard $SU(N)$ decomposition of the products of k objects in the fundamental representations (Young tableaux).

In particular, various “irreducible $SU(N)$ orbits” have been identified: they correspond to fixed-point sets invariant under the spatial rotation group. These solutions are axially symmetric and they transform according to various irreducible representations appearing in the decomposition of the direct product.

Although some of our results might be naturally expected on general grounds, a very suggestive and nontrivial aspect of our findings is the fact that the points of the vortex moduli space, describing the degenerate set of classical extended field configurations, are formally mapped

to oscillator “quantum-state” vectors, endowed with simple $SU(N)$ transformation properties. Also, the way the irreducible orbits are embedded in the full moduli space appears to be quite nontrivial, and exhibits special features of our vortex systems. For instance, an irreducible orbit associated with a definite type of Young tableau appears only once, unlike in the usual decomposition of k distinguishable objects in \mathbf{N} .

We have determined the Kähler potential on each of these irreducible orbits. Since we have used symmetry only, our Kähler potential cannot receive any quantum corrections except for the overall constant r even in non-supersymmetric theories²⁰. The results found agree with some explicit calculations made recently by some of us [29].

Extension of our considerations to more general situations in $U(N)$ theories (question of non-irreducible, general orbits in the vortex moduli space considered here, or the metric in the case of semi-local vortices, which occur when the number of flavors exceeds the number of colors [31, 32]) remains an open issue. A particularly interesting extension would however be the study of a more general class of gauge theories, such as SO, USp or exceptional groups, as the group-theoretic features of our findings would manifest themselves better in such wider testing grounds. Non-Abelian vortices were constructed in the $G' \times U(1)$ gauge theories with an arbitrary compact Lie group G' , and the orientational moduli space was found to be G'/H with some subgroup H [33]. For instance they are $SO(2N)/U(N)$ and $USp(2N)/U(N)$ in the cases of $G' = SO(2N), USp(2N)$. The SO and USp non-Abelian vortices and their moduli have been further studied in detail in the Refs. [27, 29, 34–36]. Especially, G' orbits in the moduli spaces of SO and USp non-Abelian vortices have been studied in Ref. [27]. Irreducible orbits in these cases may be classified by (skew-)symmetric Young tableaux.

Finally, a possible relation to Young tableaux for Yang-Mills instantons [37] and its application to the instanton counting [38] may be interesting. For the instanton counting, the integration over the instanton moduli space is reduced to a sum over the Young tableaux, which correspond to fixed points of the instanton moduli space under a linear combination of the $SU(N)$ action and spatial rotations, as in our case of vortices. Roughly speaking possible vortex counting should be the half of the instanton counting since Yang-Mills instantons can stably exist even in the Higgs phase when they are trapped inside non-Abelian vortices [5]. The partition function of the non-

²⁰ The renormalization group flow for r in the case of $k = 1$ vortex in $\mathcal{N} = 2$ $U(N)$ supersymmetric theories was found in Refs. [3, 4].

Abelian vortex gas was derived on a torus and a sphere in Ref. [39] by using a completely different approach of D-brane configurations and T-duality on it. A relation with such an approach and the Young tableaux for vortices developed in this paper appears to be an interesting future venue to explore.

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A Constraints on the invariants

In this appendix, we derive the constraints (2.22), (2.25) and (2.26) from the definition of the baryons

$$B_{r_1 r_2 \dots r_k}^{n_1 n_2 \dots n_k} \equiv \epsilon^{i_1 i_2 \dots i_k} Q_{i_1 r_1}^{(n_1)} Q_{i_2 r_2}^{(n_2)} \dots Q_{i_k r_k}^{(n_k)}, \quad (Q^{(n)} \equiv Z^n \psi), \quad (\text{A.1})$$

and the vortex state vector

$$|B\rangle \equiv \sum_{\substack{n_1, n_2, \dots, n_k \\ r_1, r_2, \dots, r_k}} \frac{1}{(n_1! n_2! \dots n_k!)^{\frac{1}{2}}} B_{r_1 r_2 \dots r_k}^{n_1 n_2 \dots n_k} |n_1, r_1\rangle \otimes |n_2, r_2\rangle \otimes \dots \otimes |n_k, r_k\rangle. \quad (\text{A.2})$$

1. Eq.(2.22) implies that the baryon is anti-symmetric under the exchange of any pair of indices (n, r) . This can easily be seen from the definition of the baryons

$$\begin{aligned} B_{r_1 \dots r_I \dots r_J \dots r_k}^{n_1 \dots n_I \dots n_J \dots n_k} &= \epsilon^{i_1 \dots i_I \dots i_J \dots i_k} Q_{i_1 r_1}^{(n_1)} \dots Q_{i_I r_I}^{(n_I)} \dots Q_{i_J r_J}^{(n_J)} \dots Q_{i_k r_k}^{(n_k)} \\ &= -\epsilon^{i_1 \dots i_J \dots i_I \dots i_k} Q_{i_1 r_1}^{(n_1)} \dots Q_{i_J r_J}^{(n_J)} \dots Q_{i_I r_I}^{(n_I)} \dots Q_{i_k r_k}^{(n_k)} \\ &= -B_{r_1 \dots r_J \dots r_I \dots r_k}^{n_1 \dots n_J \dots n_I \dots n_k}. \end{aligned} \quad (\text{A.3})$$

2. The annihilation operator \hat{a}_I acts on the state as

$$\begin{aligned} \hat{a}_I |B\rangle &= \sum \frac{1}{(n_1! \dots (n_I - 1)! \dots n_k!)^{\frac{1}{2}}} B_{r_1 \dots r_I \dots r_k}^{n_1 \dots n_I \dots n_k} |n_1, r_1\rangle \dots |n_I - 1, r_I\rangle \dots |n_k, r_k\rangle \\ &= \sum \frac{1}{(n_1! \dots n_I! \dots n_k!)^{\frac{1}{2}}} B_{r_1 \dots r_I \dots r_k}^{n_1 \dots n_I + 1 \dots n_k} |n_1, r_1\rangle \dots |n_I, r_I\rangle \dots |n_k, r_k\rangle. \end{aligned} \quad (\text{A.4})$$

This means that the baryon is mapped by the operator \hat{a}_I as

$$B_{r_1 \dots r_I \dots r_k}^{n_1 \dots n_I \dots n_k} \mapsto B_{r_1 \dots r_I+1 \dots r_k}^{n_1 \dots n_I+1 \dots n_k} = \epsilon^{i_1 \dots j \dots i_k} Z_{j i_I} Q_{i_1 r_1}^{(n_1)} \dots Q_{i_I r_I}^{(n_I)} \dots Q_{i_k r_k}^{(n_k)}. \quad (\text{A.5})$$

Therefore, we find that the operator $\prod_{I=1}^k (\lambda - \hat{a}_I)$ acts on the baryons as

$$\begin{aligned} B_{r_1 \dots r_k}^{n_1 \dots n_k} &\mapsto \epsilon^{j_1 j_2 \dots j_k} (\lambda \mathbf{1}_k - Z)_{j_1 i_1} \dots (\lambda \mathbf{1}_k - Z)_{j_k i_k} Q_{i_1 r_1}^{(n_1)} \dots Q_{i_k r_k}^{(n_k)} \\ &= \det(\lambda \mathbf{1}_k - Z) B_{r_1 \dots r_k}^{n_1 \dots n_k}. \end{aligned} \quad (\text{A.6})$$

Namely, the vortex state should be an eigenstate of the operator $\prod_{I=1}^k (\lambda - \hat{a}_I)$

$$\prod_{I=1}^k (\lambda - \hat{a}_I) |B\rangle = \det(\lambda \mathbf{1}_k - Z) |B\rangle. \quad (\text{A.7})$$

Comparing the coefficient of λ^i on both sides, we obtain the constraint (2.25).

3. The left hand side of Eq. (2.26) is

$$B^{A_1 \dots [A_k} B^{B_1 \dots B_k]} = \sum_{i_1, \dots, i_k} \sum_{j_1, \dots, j_k} \epsilon^{i_1 \dots i_k} \epsilon^{j_1 \dots j_k} Q_{i_1}^{A_1} \dots Q_{i_k}^{A_k} Q_{j_1}^{B_1} \dots Q_{j_k}^{B_k}, \quad (\text{A.8})$$

where A_i and B_i each denote a pair of indices (n, r) . Let us focus on the following part

$$\sum_{j_1, \dots, j_k} \epsilon^{j_1 \dots j_k} Q_{i_k}^{A_k} Q_{j_1}^{B_1} \dots Q_{j_k}^{B_k}. \quad (\text{A.9})$$

Since the indices j_1, \dots, j_k are contracted with $\epsilon^{j_1 \dots j_k}$, there exist a number I ($1 \leq I \leq k$) such that $i_k = j_I$ for each term in the sum. Therefore, all the terms in Eq. (A.9) vanish since the indices A_k and B_1, \dots, B_k are anti-symmetrized. This fact leads to the constraint Eq. (2.26).

B A toy metric on the vector space spanned by $|B\rangle$

We have not considered in the main text the metric for the vector space spanned by $|B\rangle$, introduced in Subsection 2.1, for reasons explained at the end of Subsection 2.2. Such a metric would however induce a natural metric on the vortex moduli space, which is of physical interest. For instance, one could simply assume the standard inner product $\langle B|B\rangle$; it would induce a metric specified by the following Kähler potential

$$K_{\text{toy}} = r \log \langle B|B\rangle. \quad (\text{B.1})$$

Note that the equivalence relation (2.14) is realized as Kähler transformations. Namely, the moduli space is embedded into the projective space with suitable constraints (2.26). In the case of well-separated vortices $|z_I - z_J| \gg m^{-1}$, we find that the Kähler potential (B.1) takes the form

$$K_{\text{toy}} = r \sum_{I=1}^k \left(|z_I|^2 + \log |\vec{\phi}^I|^2 \right) - r \sum_{I, J (\neq I)} \frac{|\vec{\phi}^{I\dagger} \cdot \vec{\phi}^J|^2}{|\vec{\phi}^I|^2 |\vec{\phi}^J|^2} e^{-|z_I - z_J|^2} + \dots \quad (\text{B.2})$$

The first term correctly describes free motion of k vortices while the second term describes interactions between the vortices.

Unfortunately, the interaction terms do not have the correct form; terms which behave as $1/|z_I - z_J|^2$ or $K_0(m|z_I - z_J|)$ must be present if massless or massive modes propagate between vortices, respectively. The former is the case of the Hanany-Tong metric [1] (which still does not describe the correct interactions), while the latter is the case of the correct asymptotic form obtained from the BPS equations [10].

C Metrics on $W\mathbb{C}P_{(2,1,1)}^2$ for $k = 2$ and $N = 2$

In this Appendix we will study some metrics on the intrinsic subspace $W\mathbb{C}P_{(2,1,1)}^2$ for $k = 2$ coincident vortices in the $U(2)$ gauge theory ($N = 2$). We show that two different metrics on $W\mathbb{C}P_{(2,1,1)}^2$ contain the Fubini-Study metric with the same Kähler class on $\mathbb{C}P^1$ at the diagonal edge of Fig. 3.

For any choice of metric on the moduli space, a subspace specified by a holomorphic constraint should also be a Kähler manifold. Its Kähler potential must be invariant under the global $SU(2)$ and the transformation (3.23) as

$$K_{W\mathbb{C}P^2} = rf(X) \sim r\tilde{f}(X) + \text{const.} \times \log |\phi_i|^2, \quad X \equiv \frac{|\phi_i|^4}{r|\eta|^2} \quad (\text{C.1})$$

with an arbitrary function f . For the Hanany-Tong model, $f(X)$ can be written as [18]²¹,

$$f(X) = w^2 - \log(1 - w^4), \quad w^2 = \frac{2X}{1 + X + \sqrt{1 + 6X + X^2}}. \quad (\text{C.2})$$

For the toy model (B.1) in Appendix B, $f(X)$ can be written as

$$f(X) = r \log(1 + rX). \quad (\text{C.3})$$

²¹ Here w is identical to that of Eq. (32) of the paper presented by Auzzi-Bolognesi-Shifman [18]. Actually, we can reproduce the metric Eq. (34) in their work from the above potential.

These two models have the same behavior

$$f(X) \sim \begin{cases} \log X + \text{const.}, & X \gg 1, \\ \text{const.} \times X, & X \ll 1. \end{cases} \quad (\text{C.4})$$

Since $\log X \simeq 2 \log |\phi_i|^2$, they give the usual Fubini-Study metric on $\mathbb{C}P^1$ with the same Kähler class, $2r$, for $\eta = 0$, and they have a conical singularity at $\phi_i = 0$. These features are not accidental but are guaranteed for any choice of the moduli space metric, as we show in Section 4.

D General solution of the linear constraints for $k = 3$

In this section, we consider the general solution of the linear constraints (2.22) and (2.25) for the $k = 3$ case. We have seen in Section 2.2 that the solution can be expressed by the coherent states

$$|B\rangle = \sum_{r_1, r_2, r_3} \tilde{B}_{r_1 r_2 r_3} \hat{\mathcal{A}}(|z_1, r_1\rangle \otimes |z_2, r_2\rangle \otimes |z_3, r_3\rangle). \quad (\text{D.1})$$

However, this expression is not valid globally on the moduli space since the coherent states become linearly dependent when some vortices coincide $z_i = z_j$. In order to derive a globally well-defined expression for the general solution, let us rewrite the coherent state of Eq. (D.1) as

$$\begin{aligned} |B\rangle &= \frac{1}{3!} \sum_{r_1, r_2, r_3} \sum_{\rho \in \mathfrak{S}_3} \text{sign}(\rho) \tilde{B}_{r_1 r_2 r_3} \hat{\rho} |z_1, r_1\rangle \otimes |z_2, r_2\rangle \otimes |z_3, r_3\rangle \\ &= \frac{1}{3!} \sum_{r_1, r_2, r_3} \sum_{\rho \in \mathfrak{S}_3} \text{sign}(\rho) \tilde{B}_{r_1 r_2 r_3} |z_{\rho(1)}, r_{\rho(1)}\rangle \otimes |z_{\rho(2)}, r_{\rho(2)}\rangle \otimes |z_{\rho(3)}, r_{\rho(3)}\rangle, \end{aligned} \quad (\text{D.2})$$

where $\hat{\rho}$ is an element of the symmetric group \mathfrak{S}_3 . Defining an operator \hat{v} by

$$\hat{v} \equiv \exp \left(z_1 \hat{a}_1^\dagger + z_2 \hat{a}_2^\dagger + z_3 \hat{a}_3^\dagger \right), \quad (\text{D.3})$$

and the action of the symmetric group

$$\hat{\rho} \hat{v} \hat{\rho}^{-1} \equiv \exp \left(z_1 \hat{a}_{\rho^{-1}(1)}^\dagger + z_2 \hat{a}_{\rho^{-1}(2)}^\dagger + z_3 \hat{a}_{\rho^{-1}(3)}^\dagger \right), \quad (\text{D.4})$$

we can rewrite the state $|B\rangle$ as

$$|B\rangle = \frac{1}{3!} \sum_{r_1, r_2, r_3} \sum_{\rho \in \mathfrak{S}_3} \text{sign}(\rho) \tilde{B}_{r_{\rho^{-1}(1)} r_{\rho^{-1}(2)} r_{\rho^{-1}(3)}} \hat{\rho} \hat{v} \hat{\rho}^{-1} |0, r_1\rangle \otimes |0, r_2\rangle \otimes |0, r_3\rangle. \quad (\text{D.5})$$

This means that the solution $|B\rangle$ is a linear combination of $3! = 6$ states $\hat{\rho} \hat{v} \hat{\rho}^{-1} |0, r_1\rangle \otimes |0, r_2\rangle \otimes |0, r_3\rangle$, which form a basis of the vector space of states satisfying the constraint

$$P(\hat{a}_1, \hat{a}_2, \hat{a}_3) |B\rangle = P(z_1, z_2, z_3) |B\rangle, \quad (\text{D.6})$$

for all symmetric polynomials P . However this basis is well-defined only for separated vortices since the states become degenerate when some vortices coincide. A globally well-defined basis can however be constructed as follows. Let $|S; r_1, r_2, r_3; \{z_i\}\rangle$ be the state defined by

$$|S; r_1, r_2, r_3; \{z_i\}\rangle \equiv \frac{1}{3!\Delta} \sum_{\rho \in \mathfrak{S}_3} \text{sign}(\rho) \hat{\rho} \hat{v} \hat{\rho}^{-1} |0, r_1\rangle \otimes |0, r_2\rangle \otimes |0, r_3\rangle, \quad (\text{D.7})$$

where Δ is the Vandermonde polynomial

$$\Delta(z_1, z_2, z_3) \equiv (z_1 - z_2)(z_2 - z_3)(z_3 - z_1). \quad (\text{D.8})$$

This state is a solution of the constraint (D.6) and well-defined even when the vortex centers coincide

$$|S; r_1, r_2, r_3; \{z_i\}\rangle \rightarrow \Delta(\hat{a}_1^\dagger, \hat{a}_2^\dagger, \hat{a}_3^\dagger) |0, r_1\rangle \otimes |0, r_2\rangle \otimes |0, r_3\rangle. \quad (\text{D.9})$$

The other globally well-defined solutions can be constructed by acting with polynomials of \hat{a}_i on $|S; r_1, r_2, r_3; \{z_i\}\rangle$. Note that any polynomial can be decomposed as

$$f(\hat{a}_1, \hat{a}_2, \hat{a}_3) = \sum_i g_i(\hat{a}_1, \hat{a}_2, \hat{a}_3) h_i(\hat{a}_1, \hat{a}_2, \hat{a}_3), \quad (\text{D.10})$$

where g_i 's are symmetric polynomials and h_i are polynomials satisfying

$$\langle 0, r_1| \otimes \langle 0, r_2| \otimes \langle 0, r_3| h_i(\hat{a}_1, \hat{a}_2, \hat{a}_3) P(\hat{a}_1^\dagger, \hat{a}_2^\dagger, \hat{a}_3^\dagger) = 0. \quad (\text{D.11})$$

for all symmetric polynomials P (without the constant term). Since the state $|S; r_1, r_2, r_3; \{z_i\}\rangle$ satisfies

$$g_i(\hat{a}_1, \hat{a}_2, \hat{a}_3) |S; r_1, r_2, r_3; \{z_i\}\rangle = g_i(z_1, z_2, z_3) |S; r_1, r_2, r_3; \{z_i\}\rangle, \quad (\text{D.12})$$

a symmetric polynomial $g_i(\hat{a}_1, \hat{a}_2, \hat{a}_3)$ does not create a new state. Therefore, it is sufficient to consider the polynomials $h_i(\hat{a}_1, \hat{a}_2, \hat{a}_3)$ satisfying Eq. (D.11). The space of such polynomials H is a $3! = 6$ -dimensional vector space which can be decomposed as

$$H^{(0)} \ni S, \quad (\text{D.13})$$

$$H^{(1)} \ni \tilde{Y}^1 \hat{a}_1 + \tilde{Y}^2 \hat{a}_2 + \tilde{Y}^3 \hat{a}_3, \quad (\text{D.14})$$

$$H^{(2)} \ni \tilde{X}^1 (\hat{a}_2 - \hat{a}_3)^2 + \tilde{X}^2 (\hat{a}_3 - \hat{a}_1)^2 + \tilde{X}^3 (\hat{a}_1 - \hat{a}_2)^2, \quad (\text{D.15})$$

$$H^{(3)} \ni A(\hat{a}_1 - \hat{a}_2)(\hat{a}_2 - \hat{a}_3)(\hat{a}_3 - \hat{a}_1), \quad (\text{D.16})$$

where $S, \tilde{Y}^i, \tilde{X}^i, A$ are complex numbers satisfying

$$\tilde{Y}^1 + \tilde{Y}^2 + \tilde{Y}^3 = 0, \quad \tilde{X}^1 + \tilde{X}^2 + \tilde{X}^3 = 0. \quad (\text{D.17})$$

The spaces $H^{(i)}$ are closed under the action of the symmetric group and the decomposition $H = \oplus_i H^{(i)}$ corresponds to the decomposition of the regular representation of \mathfrak{S}_3 . Acting with the elements of $H^{(i)}$ on $|S\rangle$, we obtain the following basis

$$\begin{aligned} |S\rangle &\equiv \sum_{r_1, r_2, r_3} S_{r_1 r_2 r_3} |S; r_1, r_2, r_3; \{z_i\}\rangle, \\ |Y\rangle &\equiv \sum_{r_1, r_2, r_3} (\tilde{Y}_{r_1 r_2 r_3}^1 \hat{a}_1 + \tilde{Y}_{r_1 r_2 r_3}^2 \hat{a}_2 + \tilde{Y}_{r_1 r_2 r_3}^3 \hat{a}_3) |S; r_1, r_2, r_3; \{z_i\}\rangle, \\ |X\rangle &\equiv \sum_{r_1, r_2, r_3} \left(\tilde{X}_{r_1 r_2 r_3}^1 (\hat{a}_2 - \hat{a}_3)^2 + \tilde{X}_{r_1 r_2 r_3}^2 (\hat{a}_3 - \hat{a}_1)^2 + \tilde{X}_{r_1 r_2 r_3}^3 (\hat{a}_1 - \hat{a}_2)^2 \right) |S; r_1, r_2, r_3; \{z_i\}\rangle, \\ |A\rangle &\equiv \sum_{r_1, r_2, r_3} A_{r_1 r_2 r_3} (\hat{a}_1 - \hat{a}_2)(\hat{a}_2 - \hat{a}_3)(\hat{a}_3 - \hat{a}_1) |S; r_1, r_2, r_3; \{z_i\}\rangle, \end{aligned}$$

From the anti-symmetry condition $\hat{\rho} |B\rangle = \text{sign}(\rho) |B\rangle$, we find that for all $\rho \in \mathfrak{S}_3$

$$S_{r_1 r_2 r_3} = S_{r_{\rho(1)} r_{\rho(2)} r_{\rho(3)}}, \quad (\text{D.18})$$

$$\tilde{Y}_{r_1 r_2 r_3}^i = \text{sign}(\rho) \tilde{Y}_{r_{\rho(1)} r_{\rho(2)} r_{\rho(3)}}^{\rho(i)}, \quad (\text{D.19})$$

$$\tilde{X}_{r_1 r_2 r_3}^i = \text{sign}(\rho) \tilde{X}_{r_{\rho(1)} r_{\rho(2)} r_{\rho(3)}}^{\rho(i)}, \quad (\text{D.20})$$

$$A_{r_1 r_2 r_3} = \text{sign}(\rho) A_{r_{\rho(1)} r_{\rho(2)} r_{\rho(3)}}. \quad (\text{D.21})$$

These relations imply that the tensors are in the irreducible representations of $SU(N)$. Note that in the coincident limit $z_1 = z_2 = z_3$, these states reduce to

$$\begin{aligned} |S\rangle &\rightarrow \sum_{r_1, r_2, r_3} S_{r_1 r_2 r_3} (\hat{a}_1^\dagger - \hat{a}_2^\dagger)(\hat{a}_2^\dagger - \hat{a}_3^\dagger)(\hat{a}_3^\dagger - \hat{a}_1^\dagger) |0, r_1, r_2, r_3\rangle, \\ |Y\rangle &\rightarrow \sum_{r_1, r_2, r_3} (Y_{r_1 r_2 r_3}^1 (\hat{a}_2^\dagger - \hat{a}_3^\dagger)^2 + Y_{r_1 r_2 r_3}^2 (\hat{a}_3^\dagger - \hat{a}_1^\dagger)^2 + Y_{r_1 r_2 r_3}^3 (\hat{a}_1^\dagger - \hat{a}_2^\dagger)^2) |0, r_1, r_2, r_3\rangle, \\ |X\rangle &\rightarrow \sum_{r_1, r_2, r_3} (X_{r_1 r_2 r_3}^1 \hat{a}_1^\dagger + X_{r_1 r_2 r_3}^2 \hat{a}_2^\dagger + X_{r_1 r_2 r_3}^3 \hat{a}_3^\dagger) |0, r_1, r_2, r_3\rangle, \\ |A\rangle &\rightarrow \sum_{r_1, r_2, r_3} A_{r_1 r_2 r_3} |0, r_1, r_2, r_3\rangle, \end{aligned}$$

where $|0, r_1, r_2, r_3\rangle = |0, r_1\rangle \otimes |0, r_2\rangle \otimes |0, r_3\rangle$, and

$$\begin{aligned} Y^1 &\equiv \tilde{Y}^2 - \tilde{Y}^3, & Y^2 &\equiv \tilde{Y}^3 - \tilde{Y}^1, & Y^3 &\equiv \tilde{Y}^1 - \tilde{Y}^2; \\ X^1 &\equiv -6(\tilde{X}^2 - \tilde{X}^3), & X^2 &\equiv -6(\tilde{X}^3 - \tilde{X}^1), & X^3 &\equiv -6(\tilde{X}^1 - \tilde{X}^2). \end{aligned} \quad (\text{D.22})$$

By rewriting the solution (D.5) as a linear combination of these states, we obtain the globally well-defined general solution to the linear constraints.

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